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Mathematical 3-D Model for Whiplash Simulation

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Introduction

Several years ago, the author developed a mathematical 2-D model to quantitatively describe the forces, moments, and motions of vehicles and dummy occupants during collisions. The equations of motion and applicable logic statements were programmed in Basic to use on PC's (IBM compatible personal computers). Simulations of many crashes have been successfully run on PC's using the 2-D model. Clients are usually interested in the responses of the dummy head and torso and the calculated magnitudes of forces and moments in the neck joint. These data are particularly useful in litigation cases where medical findings are subjective (e.g. soft tissue injuries) and opinions of the treating and adverse health care professionals differ significantly.

When additional relevant information has been received, the model has been refined. The model has been modified to accommodate crashes in which the target and bullet vehicles are off-center and reasonably close in axial alignment. A description of the 2-D model was presented at the 1993 Annual Meeting of the American Academy of Forensic Sciences at San Antonio, Texas.

In a particular recent case, the nature of the collision and the posture of the occupant driver are not within the inherent constraints of the 2-D model. In order to adequately study this case, a mathematical 3-D model would be needed.

The adage that "necessity is the mother of invention" applies to an unfunded project conducted by the author. This paper describes a mathematical 3-D model developed by the author for simulating vehicle and occupant responses during collisions.

Methodologies

State-Space Formulation

In this formulation, the time derivatives of all system variables are described as a function of the current values of those variables and as a function of external input(s) to the system. The general vector-matrix form is

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$$|\dot{X}| = f_1|X| + f_2|U|$$

where $|X| = \begin{vmatrix} X_1 \\ . \\ . \\ . \\ X_n \end{vmatrix}$, $|U| = \begin{vmatrix} U_1 \\ . \\ . \\ . \\ U_m \end{vmatrix}$

f_1, f_2 are known mathematical functions

Second order derivatives (e.g. accelerations) are treated by defining a velocity vector and noting that velocity is the time derivative of displacement and that acceleration is the time derivative of velocity;

$$|V| = |\dot{X}|, \text{ and } |A| = |\dot{V}|$$

Lumped Parameters

A distributive object is represented as a series of rigid masses interconnected by springs and dampers. The motion of each lump of mass is defined by a linear translation of its center of mass in 3-D space and by rotation about its center of mass. In this manner, complex objects are approximated by more familiar vibration systems.

Linear translation spring forces between adjacent lumps of mass are defined by the stretch of the spring in each of the three directional coordinates, x, y, and z

$$F_{sx1\text{-}2} = -K_x(X_1-X_2),\ F_{sy1\text{-}2} = -K_y(Y_1-Y_2),\ F_{sz1\text{-}2} = -K_z(Z_1-Z_2)$$

where **F_s = spring restoring force - lb_f**
K = spring constant - lb_f/in
X, Y, Z are directional coordinates - inches

Linear translation damper forces between adjacent lumps of mass are defined by the velocity differential across the damper

$$F_{dx1\text{-}2} = -C_x(\dot{X}_1-\dot{X}_2),\ F_{dy1\text{-}2} = -C_y(\dot{Y}_1-\dot{Y}_2),\ F_{dz1\text{-}2} = -C_z(\dot{Z}_1-\dot{Z}_2)$$

where **F_d = damper resistance force - lb_f**
C = damper constant - lb_f/in/sec
$\dot{X}$, $\dot{Y}$, $\dot{Z}$ are derivatives of coordinates - in/sec

Linear torsional spring and damper torques between adjacent lumps of mass are represented by equations similar to the preceding two equations.

$$T_{s\O z} = -K_{t\O z}(\O_{z1}-\O_{z2}),\ T_{s\O x} = -K_{t\O x}(\O_{x1}-\O_{x2}),\ T_{s\O y} = -K_{t\O y}(\O_{y1}-\O_{y2})$$

where T_s = torsional spring restoring torque - in lb_f
K_t = torsional spring constant - in lb_f/rad
$\O_z$, $\O_x$, $\O_y$ are rotational coordinates - radians

and

$$T_{d\O z} = -C_{t\O z}(\dot{\O}_{z1}-\dot{\O}_{z2}),\ T_{d\O x} = -C_{t\O x}(\dot{\O}_{x1}-\dot{\O}_{x2}),\ T_{d\O y} = -C_{t\O y}(\dot{\O}_{y1}-\dot{\O}_{y2})$$

where T_d = torsional damper resistance torque - in lb_f
C_t = torsional damper constant - in lb_f/rad/sec
$\dot{\O}_z$, $\dot{\O}_x$, $\dot{\O}_y$ are derivatives of coordinates - rad/sec

When the spring and damping constants are not known, experience and intuition will usually provide at least approximate values for natural frequencies and damping ratios. It can be shown that the spring constant is related to the natural frequency and mass by

$$K = \frac{W}{g}(2\pi F_n)^2$$

where W = weight of the mass - lb
g = 386.088 in/sec/sec at 45 deg lat and sea level
F_n = natural frequency - hertz
π = 3.141593 to 7 significant figures

and that the damping constant is related to the natural frequency, the damping ratio, and the mass by

$$K = \frac{W}{g}(2\pi F_n)D_r$$

where D_r = damping ratio - decimal fraction of critical

The torsional spring constant is related to the natural frequency and moment of inertia by

$$K_t = \frac{W}{g}\bar{R}^2(2\pi F_{nt})^2$$

where K_t = torsional spring constant - in lbf/rad

$\bar{R}$ = radius of gyration - in
F_{nt} = torsional natural frequency - hertz

and the rotational damping constant is related to the natural frequency, the damping ratio, and the moment of inertia by

$$C_t = 2 \frac{W}{g} \bar{R}^2 (2 \pi F_{nt}) D_{rt}$$

where **C_t = torsional damping constant - in lb_f/rad/sec**
D_{rt} = torsional damping ratio - fraction of crit.

Programmable on a PC

Over the past years, PC's (IBM compatible personal computers) have become more affordable, more readily available, and have had great improvements in their computing capabilities. Besides that, the author has several available for usage. Therefore, the new mathematical 3-D models are programmed for PC's on which the simulations are run.

Numerical Integration

Since the simulations are performed with digital computers, integrations must be performed using numerical methods. The Euler method is perhaps the simplest to use (and subject to the greatest inaccuracies). In this method the value of a variable at the next time step is related to its current value and the current value of its derivative by

$$X_i(t + \Delta t) = X_i(t) + \dot{X}_i(t)\, \Delta t$$

where **X_i = variable**
$\dot{X}_i$ = time derivative of the variable
t = current time - sec
Δt = time step - sec

The improved Euler method, also called the Heun method, uses an average value for the derivative as

$$X_i(t + \Delta t) = X_i(t) + \frac{\dot{X}_i(t) + \dot{X}_i(t + \Delta t)}{2} \Delta t$$

Other methods are available which use weighted averages or higher orders of derivatives. The Runge-Kutta method is one such example.Current versions of the 3-D mathematical model use the Euler or Heun methods with small time steps to minimize possible inaccuracies. (e.g. Δt = 0.001 sec.)

Newton's Laws of Motion

The equations of motion are taken to be translation of the center mass for each lump of mass in inertial space (earth coordinates) and rotations about its center of mass (body coordinates). The function f_i in the state-space formulation involves calculations of all external forces and torques acting on each lump of mass within the system.

When the forces are expressed in earth coordinates, the equations have the familiar forms

$$\sum F_{xe} = \frac{W}{g} \dot{V}_{xe}\ ,\ \sum F_{ye} = \frac{W}{g} \dot{V}_{ye}\ ,\ \text{and}\ \sum F_{ze} = \frac{W}{g} \dot{V}_{ze}$$

where F_{xe}, F_{ye}, F_{ze} **are forces in earth axes - lb$_f$**

$V_{xe} = \dot{X}_e$, $V_{ye} = \dot{Y}_e$, **and** $V_{ze} = \dot{Z}_e$ **- in/sec**

When the torques are expressed in body coordinates, the equations have cross-coupling terms and are written as

$$\sum T_{zb} = I_{zb} \dot{V}_{\text{Ø}zb} + (I_{yb} - I_{xb})\ \dot{\text{Ø}}_{yb}\ \dot{\text{Ø}}_{xb}\ ,$$

$$\sum T_{xb} = I_{xb} \dot{V}_{\text{Ø}xb} + (I_{zb} - I_{yb})\ \dot{\text{Ø}}_{zb}\ \dot{\text{Ø}}_{yb}\ ,$$

and

$$\sum T_{yb} = I_{yb} \dot{V}_{\text{Ø}yb} + (I_{xb} - I_{zb})\ \dot{\text{Ø}}_{xb}\ \dot{\text{Ø}}_{zb}$$

where T_{zb}, T_{xb}, T_{yb} **are torques about body axes - in lb$_f$**

I_{zb}, I_{xb}, I_{yb} **are moments of inertia about body axes**

$V_{\text{Ø}zb} = \dot{\text{Ø}}_{zb}$, $V_{\text{Ø}xb} = \dot{\text{Ø}}_{xb}$, **and** $V_{\text{Ø}yb} = \dot{\text{Ø}}_{yb}$ **- rad/sec**

Coordinate Transformation Equations

It is usually easier to express some of the forces and most of the torques in terms of body coordinates rather than in terms of earth coordinates. In the previous section of this paper the force equations are in earth coordinates and the torque equations are in body coordinates. Solutions to these equations are facilitated by the development of additional equations which can be used to transform variables from one coordinate system to the other.

The author derived transformation equations for the following order of rotation: 1. about body z-axis (yaw), 2. about body x-axis (roll), and 3. about body y-axis (pitch).

Earth to Body

$$\begin{vmatrix} X_b \\ Y_b \\ Z_b \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} X_e - X_{cg} \\ Y_e - Y_{cg} \\ Z_e - Z_{cg} \end{vmatrix}$$

where **X_b, Y_b, Z_b = position vector in body coordinates**
X_e, Y_e, Z_e = position vector in earth coordinates
X_{cg}, Y_{cg}, Z_{cg} = center of mass in earth coordinates

from matrix arithmetic

$$X_b = a_{11}(X_e - X_{cg}) + a_{12}(Y_e - Y_{cg}) + a_{13}(Z_e - Z_{cg}),$$
$$Y_b = a_{21}(X_e - X_{cg}) + a_{22}(Y_e - Y_{cg}) + a_{23}(Z_e - Z_{cg}),$$
and $$Z_b = a_{31}(X_e - X_{cg}) + a_{32}(Y_e - Y_{cg}) + a_{33}(Z_e - Z_{cg})$$

Body to Earth

$$\begin{vmatrix} X_e \\ Y_e \\ Z_e \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \begin{vmatrix} X_b \\ Y_b \\ Z_b \end{vmatrix} + \begin{vmatrix} X_{cg} \\ Y_{cg} \\ Z_{cg} \end{vmatrix}$$

from matrix arithmetic

$$X_e = a_{11} X_b + a_{21} Y_b + a_{31} Z_b + X_{cg},$$
$$Y_e = a_{12} X_b + a_{22} Y_b + a_{32} Z_b + Y_{cg},$$
and $$Z_e = a_{13} X_b + a_{23} Y_b + a_{33} Z_b + Z_{cg}$$

The matrix elements in the preceding vector-matrix transformation equations are

$$a_{11} = \cos Ø_y \cos Ø_z - \sin Ø_y \sin Ø_x \sin Ø_z$$
$$a_{12} = \cos Ø_y \sin Ø_z + \sin Ø_y \sin Ø_x \cos Ø_z$$
$$a_{13} = - \sin Ø_y \cos Ø_x$$
$$a_{21} = - \cos Ø_x \sin Ø_z$$
$$a_{22} = \cos Ø_x \cos Ø_z$$
$$a_{23} = \sin Ø_x$$
$$a_{31} = \sin Ø_y \cos Ø_z + \cos Ø_y \sin Ø_x \sin Ø_z$$
$$a_{32} = \sin Ø_y \sin Ø_z - \cos Ø_y \sin Ø_x \cos Ø_z$$
$$a_{33} = \cos Ø_y \cos Ø_x$$

Caution: Since matrix multiplications are not associative or commutative, the resulting equations will be different for a different order of rotation. Therefore, it is important to maintain the proper order of rotation within the algorithm.

Selection of Coordinate Systems

The equations derived to this point assume orthogonal axes which are mutually perpendicular and satisfy the "right hand rule". The author chooses earth coordinates such that the x-axis is horizontal and parallel to the length of the roadway with the origin at a convenient reference point and x being positive progressing in the direction of travel of the selected driving lane. The y-axis is horizontal and perpendicular to the length of the roadway with the origin on the right fogline of the selected driving lane and y being positive progressing from the fogline toward the centerline. The z-axis is vertical with the origin on the road surface and z being positive progressing vertically upward from the road surface.

The SAE (Society of Automotive Engineers) coordinate convention has the origin on the centerline with y positive toward the right fogline and z positive downward. During most of the author's inspections of accident scenes, traffic conditions have forced most measurements to be made from the road shoulder rather from the centerline. Therefore, field notes usually have y measurements recorded from the fogline. For this reason and for a personal preference for z being positive upward as in the Autocad software, the author chooses to not use the SAE coordinate convention.

With the author's selection of coordinate convention, $\emptyset_z$ (yaw) is positive ccw as viewed from above, $\emptyset_x$ (roll) is positive rotating upward from the selected driving lane surface, and $\emptyset_y$ (pitch) is positive when the leading end rotates downward. The body coordinates for each lump of mass are chosen to have the same sense as the earth coordinates when the vehicle is following the roadway with its occupant(s) seated and facing straight ahead in the direction of travel.

Mathematical Model Schematics

In any lumped parameter representation of objects, there are necessarily only a finite number of degrees of freedom for any simulation. This number can be increased by increasing the number of lumps of mass. Since increasing the number of lumps of mass adds to the model complexity and increases the computer requirements, compromises must be made. Current versions of the mathematical model are defined schematically in this section.

Vehicles

Each vehicle is represented as two lumps of mass and a massless bumper. The bumper is connected to the first mass (approximately 25% of the vehicle sprung mass) by springs and dampers. The first mass is connected to the second mass (approximately 75% of the sprung mass) by additional springs and dampers. The seat bottom, seat back, and seat belts are represented as deformable but anchored to the second mass. Each mass translates and rotates in 3-D constrained by interactions with the road surface and with the other lumped masses.

Vehicle Schematic #1, #2, and #3 are top, right side, and front end views respectively demonstrating the lumped parameter representation for each vehicle.

Occupants

Occupant dummies in each vehicle are represented as 3, 4, or 5 lumps of mass. All representations have separate masses representing the head, torso, and abdomen. The head (approximately 7% of the dummy body mass) is connected to the torso by springs and dampers. The torso (approximately 43% of the dummy body mass) is also connected to the abdomen by springs and dampers and it is acted upon by the seat back and shoulder belt. The abdomen is acted upon by the seat bottom, seat back, and lap belt. In the 4 and 5 lump models, additional lumps of mass represent the legs (combined in the 4 lump model or individually in the 5 lump model) connected to the abdomen by springs and dampers and acted upon by the seat bottom.

Occupant and Seat Schematic #1, #2, and #3 are top, right side, and front views respectively demonstrating the lumped parameter representation for an occupant dummy. These drawings represent a 4 lump model.

Conclusions

A mathematical 3-D model has been developed which is the basis for computer simulations of "whiplash" during vehicle collisions. The basic equations used in the formulations are presented. Simulation and case study results are the subject for another paper.

Nomenclature

$|A| = |\dot{V}|$ = Acceleration Vector (velocity derivative)
A_n = nth element of acceleration vector
A_x, A_y, A_z - inches/second/second
α_z, α_x, α_y - radians/second/second

$a_{ij\ (i = 1, 2, 3;\ j = 1, 2, 3)}$ = Transformation matrix elements

$_b$ = Subscript for body coordinates

C = Damper constant
linear elements - pound*second/inch
rotational elements - inch*pound*second/radian

D_r = Damping ratio - decimal fraction of critical value

$_d$ = Subscript for damper forces and torques

$_e$ = Subscript for earth coordinates

F_n = Natural frequency - hertz (cycles per second)

F_{xb}, F_{yb}, F_{zb} = Force components in body axes - pounds

F_{xe}, F_{ye}, F_{ze} = Force components in earth axes - pounds

f_1, f_2 = Expressible mathematical functions

g = Gravity constant = 386.088 inches/second/second at 45 degrees latitude and sea level

I_{zb}, I_{xb}, I_{yb} = Moments of inertia about body axes - pound*inch*second*second

K = Spring Constant
linear elements - pound/inch
rotational elements = inch*pound/radian

lb_f = Pounds force

$\bar{R}$ = Radius of gyration - inches

$_s$ = Subscript for spring forces and torques

T_{zb}, T_{xb}, T_{yb} = Torques about body axes - inch*pound

t = Time - seconds

Δt = Time step for numerical integration - second

|U| = Control (external input) vector
Um = mth element of control vector

$|V| = |\dot{X}|$ = Velocity vector (position derivative)
Vn = nth element of velocity vector
V_x, V_y, V_z - inches/second
ω_z, ω_x, ω_y - radians/second

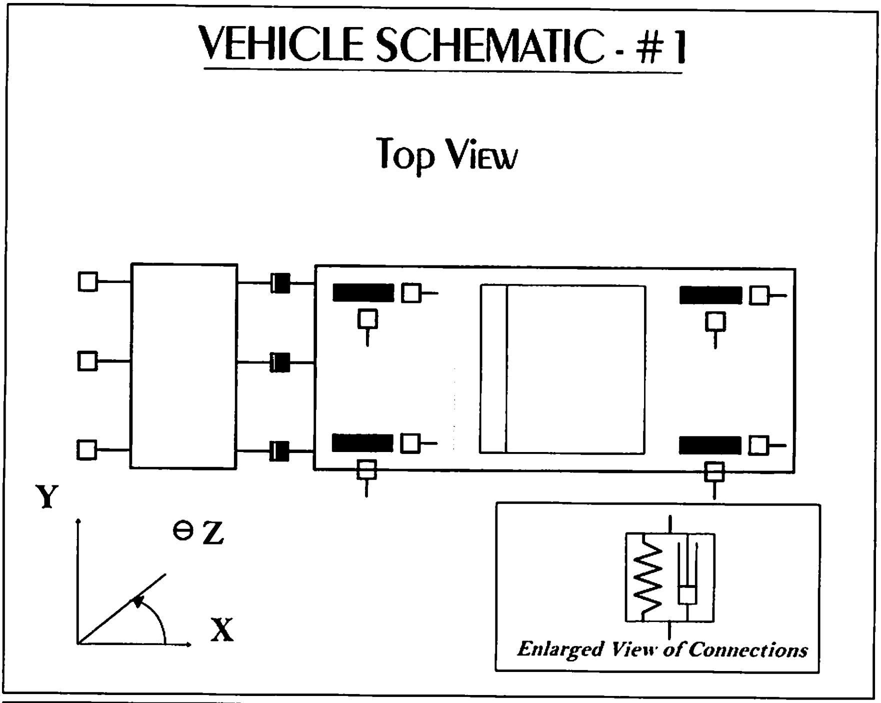
|X| = State (position) vector
Xn = nth element of state vector
X, Y, Z - inches from origin
$\O_z$, $\O_x$, $\O_y$ - radians from straight ahead level
e.g. Xn = x, y, z, $\O_z$, $\O_x$, $\O_y$, v_x, v_y, v_z, w_z, w_x, w_y
for each lump of mass

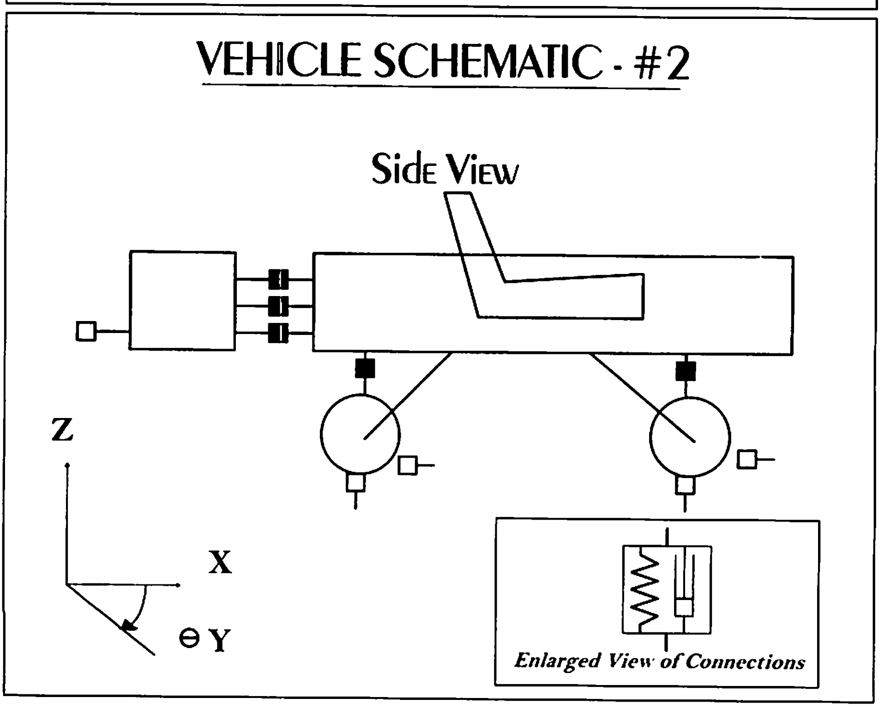
α_z (yaw), α_x (roll), α_y (pitch) = Angular accelerations - radians/second/second

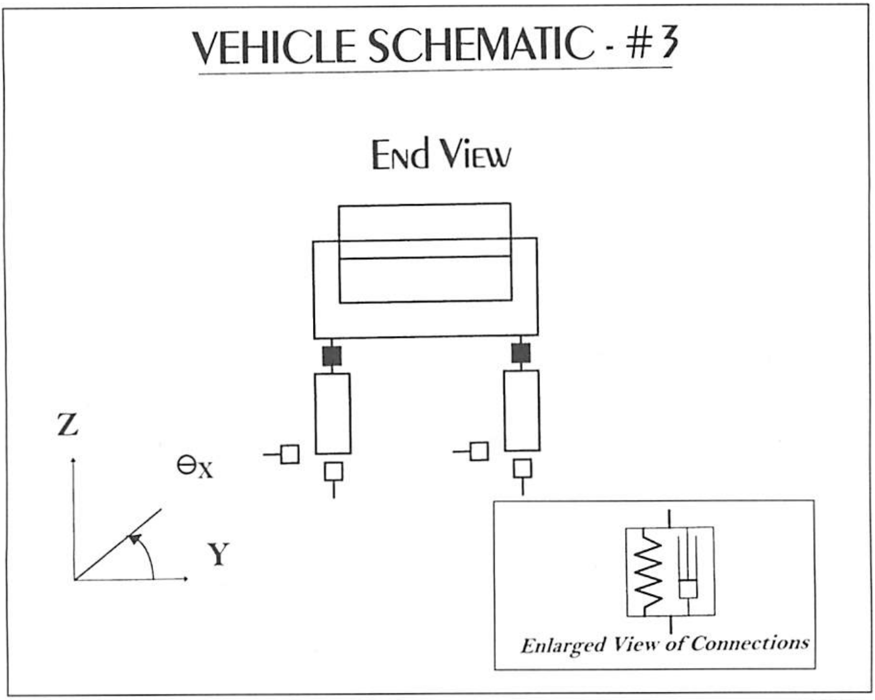
ω_z (yaw), ω_x (roll), ω_y (pitch) = Angular velocities - radians/second

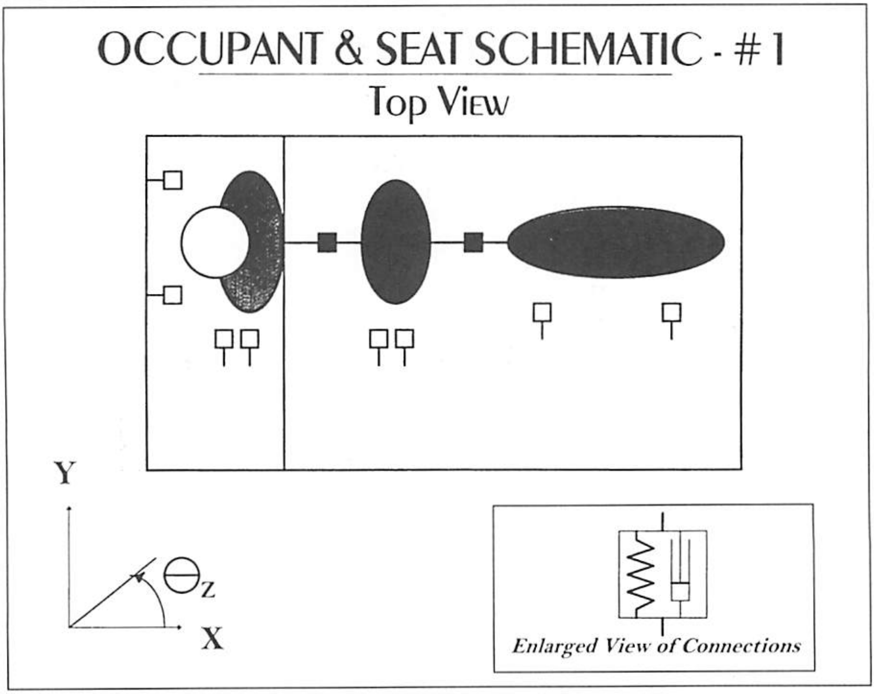
$\O_z$ (yaw), $\O_x$ (roll), $\O_y$ (pitch) = Angular rotations - radians

π = 3.141593 to seven significant figures


VEHICLE SCHEMATIC - #1
Top View
Y
θ Z
X
Enlarged View of Connections

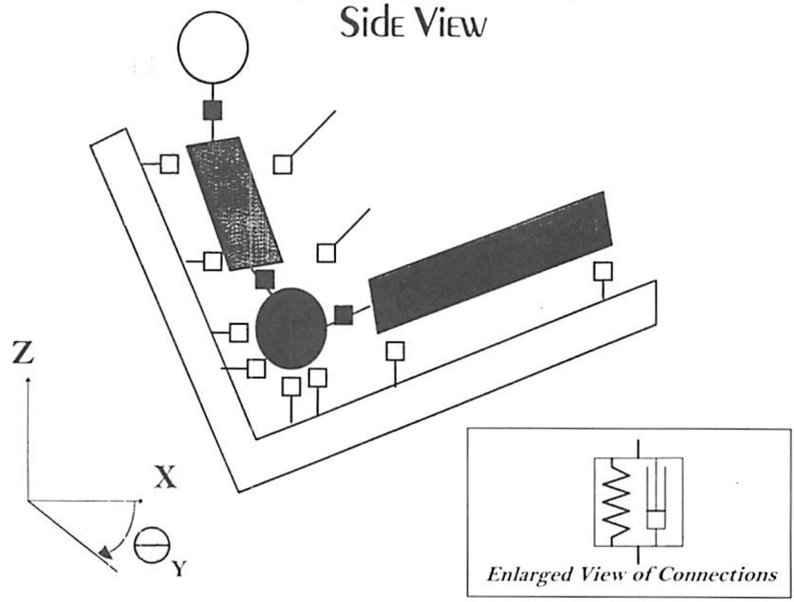

VEHICLE SCHEMATIC - #2
Side View
Z
X
θ Y
Enlarged View of Connections


VEHICLE SCHEMATIC - #3
End View
Z
ΘX
Y
Enlarged View of Connections


OCCUPANT & SEAT SCHEMATIC - #1
Top View
Y
ΘZ
X
Enlarged View of Connections

OCCUPANT & SEAT SCHEMATIC - #2

Side View



OCCUPANT & SEAT SCHEMATIC - #3

Front View