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3-D Math Model for Vehicle Dynamics Simulations Including Effects of Tires, Suspensions, and Terrain

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Abstract

A young man who was the left back seat passenger in a sedan traveling southbound in a series of downhill curves suffered severe head injuries after the driver lost control and crossed the centerline into the path of an oncoming northbound sedan. The front of the northbound sedan collided with the right side of the southbound sedan at an oblique angle. The young man was using his available seat belt, which consisted of a lap belt only without a shoulder belt.

Investigation revealed that the same model automobile was supplied with shoulder belts for rear seat passengers when the automobiles were sold in certain European countries. It was also determined that this particular vehicle had a mounting bracket for a shoulder belt already installed but hidden behind the roof liner. Naturally, the attorney representing the young man raised the question "If a shoulder belt had been available in this particular vehicle, would the young man's injuries have been less severe?" Intuition alone would not be sufficient to answer this question. Accurate descriptions of vehicle motions would be needed, particularly for the southbound sedan.

Other researchers had previously developed mathematical models for analyzing vehicle motions using 2-D and quasi 3-D formulations. Published reviews of those models indicated that they would probably not be sufficient for this case. A 3-D mathematical model would be needed which includes effects of tires, suspensions, and terrain. Such a model was developed and programmed for computer simulations of vehicle dynamics and motions. The model accommodates a wide range of initial conditions and is described in this paper.

Accident Investigations

On the night of the accident, local law enforcement officials investigated the accident and photographed the scene before the vehicles had been moved. On the following days, photographs were taken in daylight showing various details upon which preliminary accident reconstruction analyses would be based.

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Approximately 21 months after the accident with these photographs in hand, the authors conducted detailed inspections of the accident scene and recreated both vehicle final rest positions and certain tire marks attributed to the southbound sedan. Pertinent measurements were taken and recorded graphically. In the interim; the authors inspected, measured, and photographed both vehicles. A few selected photographs are included in this paper.

Figure 1 shows the accident scene looking in the direction of travel for the downhill southbound sedan. The final rest position of the southbound sedan is on the grassy incline beyond the sidewalk on the left side of the roadway near the middle of this photograph. The final rest position of the northbound sedan is on the uphill lanes of the roadway near where the southbound sedan hopped the curb.

Figures 2 and 3 are left rear and front views respectively of the northbound sedan. The front bumper is displaced from the driver's side to the passenger side suggesting that the principal direction of force (PDOF) was also from the driver's side to the passenger side.

Figures 4 and 5 are left side and right rear views respectively of the southbound sedan. The impact involves the right rear wheel and fender and rear portions of the right rear passenger door. The shape of the crushed area alone can easily lead one to believe that the PDOF was directed toward the left front and hence to conclude that a shoulder belt would not have been much help to the left rear seat passenger.

Detailed inspections of the southbound sedan by the authors revealed a moderate sized dent in the frame of the right front passenger seat backrest at its left rear corner approximately 2/3 of the way up from the seat bottom junction. Dents were also noted in the interiors of the right front and right rear passenger doors, both of which were forward and to the right from the normal sitting positions of the passengers in those seats. These observations suggested that all three passengers moved in a direction forward and approximately 50 degrees to the right relative to the vehicle before impacting against interior components. Furthermore, it appeared that the left back seat passenger impacted his head against the front passenger seat backrest. These results suggest that the left back seat passenger would have benefited substantially from a shoulder belt. In order to answer the attorney's question, the obvious discrepancy between the results here and in the previous paragraph must be resolved.

Figure 6 is a plan view scale drawing showing the final vehicle rest positions and tire marks left by the southbound sedan. The features here are based on photographs and measurements taken by investigating officers and upon the authors' detailed inspections and measurements of the accident scene.

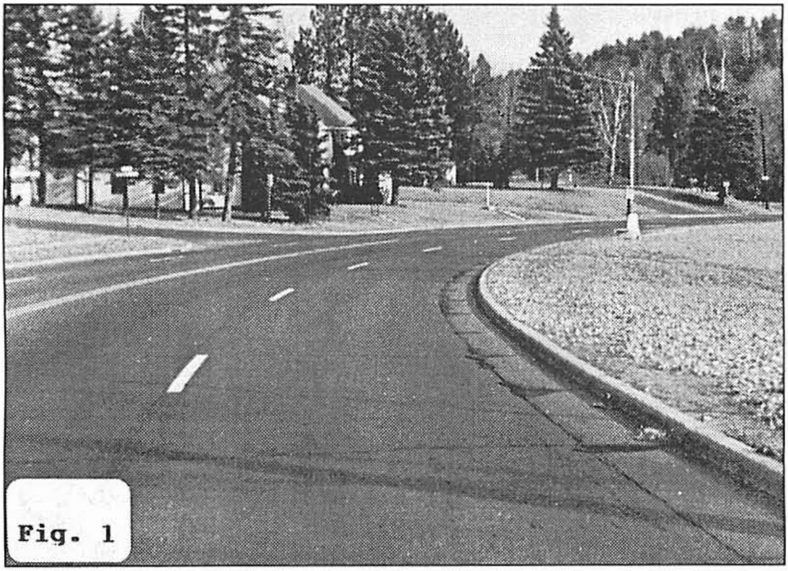


Fig. 1

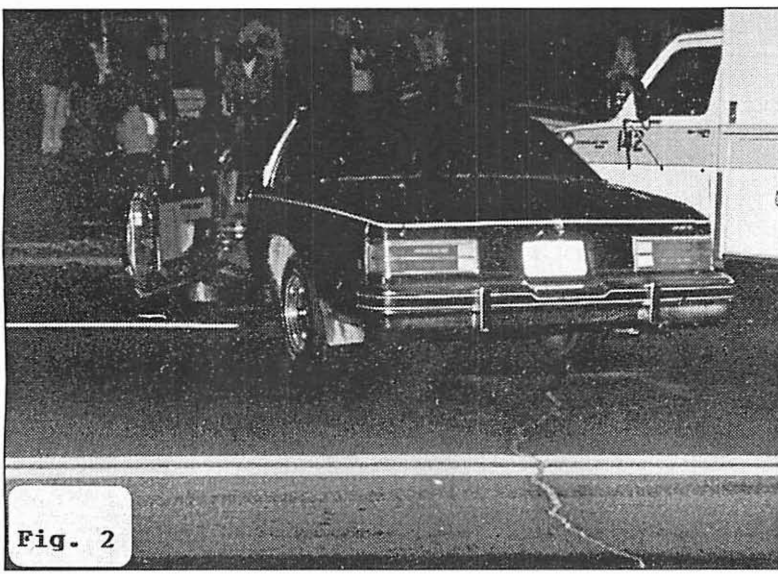


Fig. 2

Tire marks establish the pre-collision and post-collision paths of the southbound sedan and the post-collision path of the northbound sedan. Standard accident reconstruction analyses methods used by the authors established approximate vehicle speeds which indicate that the northbound sedan was not exceeding the speed limit and that the southbound sedan was exceeding the speed limit by as much as 10 mph. (A non-engineer accident reconstructionist

Fig. 3

Fig. 4

had misused the critical speed formula and rendered an opinion that this speed exceeded the speed limit by 33 mph.)

This case involved several medical and personal injury damages issues outside the scope of the mechanical engineering backgrounds of the authors. Dr. Richard W. McLay, PE (M535) with a bioengineering background participated

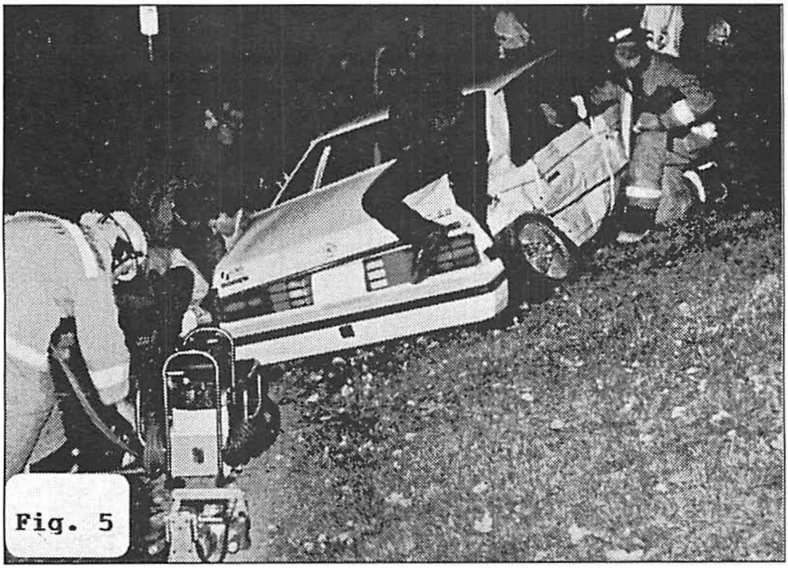


Fig. 5

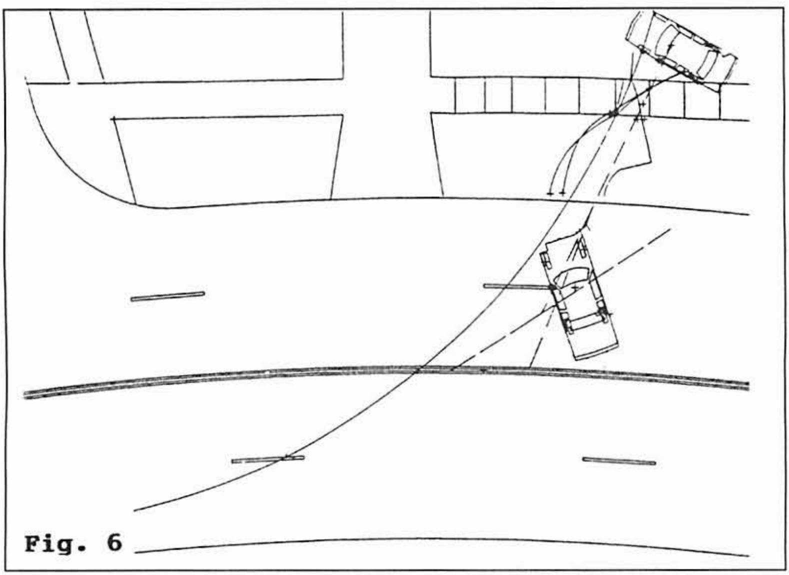


Fig. 6

in some of the inspections and consultations with the authors and bridged the gaps between the authors and the medical experts. Several liability issues were developed and addressed including the following:

1. Conduct of each driver including travel speeds and evasive actions

2. Conduct of the landscape contractor in dropping a roll of sod on the road

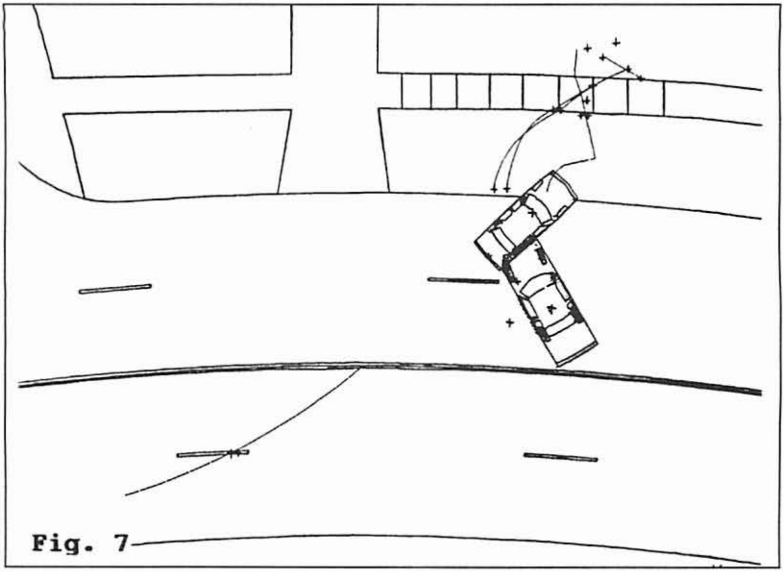


Fig. 7

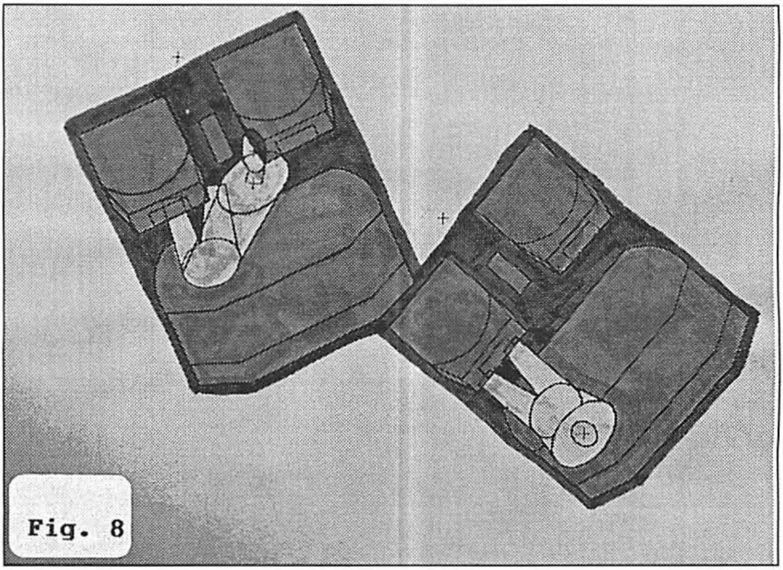


Fig. 8

near where the downhill sedan began to lose control

3. Conduct of city officials in permitting the sod to remain for several days

4. Conduct of the vehicle manufacturer in not providing rear seat shoulder belts

5. Contributions of 1, 2, and 3 to the cause of the accident
6. Contribution of 4 to enhancement of injury severity, particularly to the head injuries suffered by the left back seat passenger

Mathematical Model Development

It became apparent that the controversies involving item 6 could be resolved by an accurate determination of the time histories for the translational and rotational accelerations, velocities, and positions of the southbound sedan. These time histories would also establish and/or confirm controversies involving item 1. A mathematical model was developed for 3-dimensional vehicle dynamics simulations which provide the desired time histories. In the development of this model a small portion of the effort was funded by the client and the remainder was unfunded.

The model includes effects of tires, suspensions, and terrain. It is based on a lumped parameter method, a state-space formulation, and numerical integration techniques. Newton's laws of motion in complete 3 dimensions and applicable logic statements are programmed in the computer language Basic. Using initial values and other data defined and input by the user, a computer performs a simulation of the vehicle dynamic responses, including effects of wheel hop, according to the model commanded calculations and operations. Output of selected parameters is printed in tabular form on the monitor screen or on paper. If desired, output could also be sent to a computer file. For a more detailed description of the mathematical model, see Appendix A.

The results are simulations of dynamic responses to initial value problems; they are not closed form solutions to boundary value problems. Therefore, in general several iterations are needed to provide a desired solution that matches the accident data. Since the model by itself does not assure uniqueness of a solution, the user must evaluate whether any other feasible solutions exist.

Results

The mathematical model driven computer program was exercised to simulate pre-collision and post collision dynamics individually of the southbound sedan and the northbound sedan. These results combined with those from separate analyses of collision dynamics gave a complete solution.

1. Figure 7 is a plan view scale drawing showing the vehicle positions at the instant of separation following the impact.
2. At initial impact, the northbound sedan speed was approximately 15 mph and the southbound sedan speed was approximately 30 mph. When the

southbound sedan swerved in an apparent attempt to avoid the sod roll in the right southbound lane, its speed was approximately 37 mph.

3. Figure 8 is a plan view scale drawing showing a simulated dummy sitting normally in the left back passenger seat of the southbound sedan immediately prior to impact and at the completion of impact.

4. In the solution demonstrated here, the PDOF on the southbound sedan is directed toward the left rear at or near the right rear wheel (which is not obvious from its crush in Figure 5). The PDOF on the northbound sedan is directed from left to right at the front end which is consistent with its crush in Figure 5.

5. The left back seat passenger in the southbound sedan would not have suffered severe head injuries if he had a shoulder belt because it would have prevented the secondary impact of his head against the front seat backrests.

6. These results are stated with confidence because the simulations performed by the mathematical model driven computer program are consistent with observed data and establish the PDOF of the southbound sedan.

Conclusions

A mathematical model driven computer program has been developed which simulates vehicle dynamic responses in complete 3-dimensions including effects of tire steering, skidding, braking, and contact patch spring/damper forces; suspension springs, shock absorbers, anti-roll elements, and anti-dive links; and terrain variations in elevation and in drag factor. Each wheel is treated separately and individually, and the effects of unsprung and sprung masses are included.

The model was developed initially to address issues raised in the specific case as described in this paper. It has also been used in other cases involving one or more of the following: post collision run out, pre-collision trajectories, partial vaulting, full vaulting, and non-collision handling characteristics.

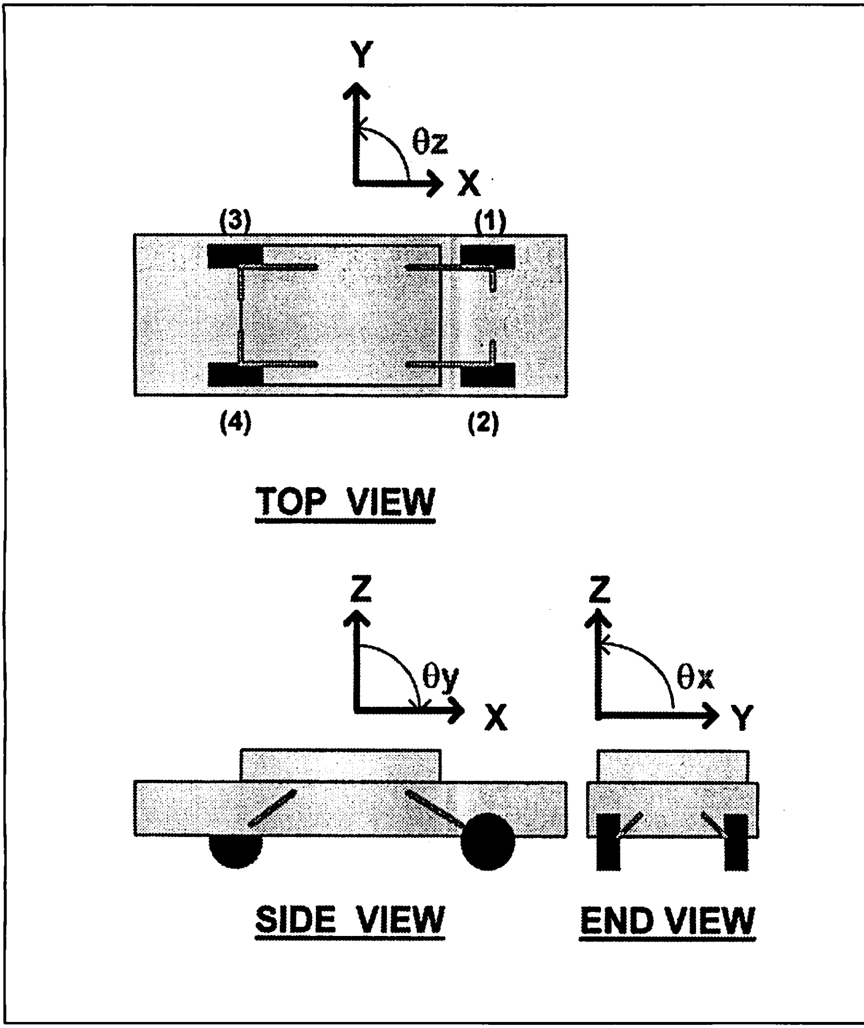
Appendix A

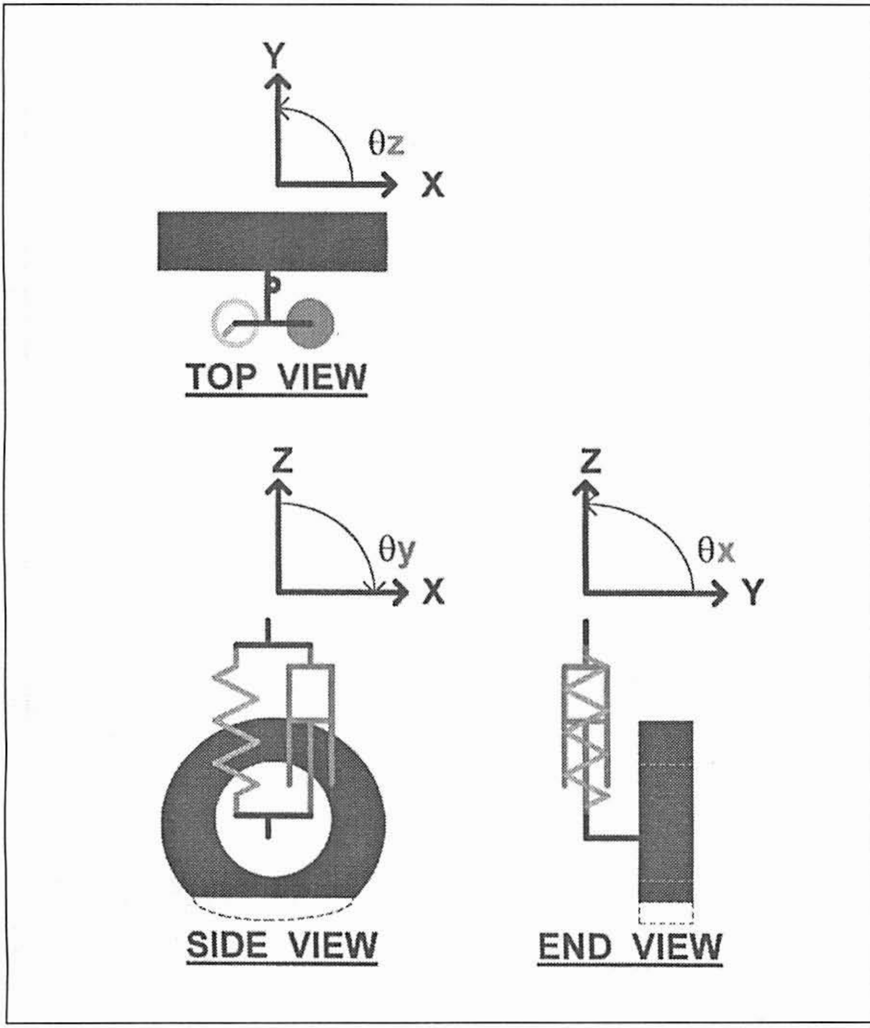
Mathematical Model Description

Including

Schematic Drawings,

Equations and Nomenclature


Y
θz
X
(3)
(1)
(4)
(2)
TOP VIEW
Z
θy
X
Z
θx
Y
SIDE VIEW
END VIEW


Y
θz
X
TOP VIEW
Z
θy
X
SIDE VIEW
Z
θx
Y
END VIEW

EQUATIONS of MOTION:

Translation of c.g. in Earth Coordinates

$$\Sigma Fx = M \; Ax$$

$$\Sigma Fy = M \; Ay$$

$$\Sigma Fz = M \; Az$$

Rotation about c.g. in Vehicle Coordinates

$$\Sigma Tzc = Izc \, \alpha zc + (Iyc - Ixc) \, \omega yc \, \omega xc$$

$$\Sigma Txc = Ixc \, \alpha xc + (Izc - Iyc) \, \omega zc \, \omega yc$$

$$\Sigma Tyc = Iyc \, \alpha yc + (Ixc - Izc) \, \omega xc \, \omega zc$$

EQUATIONS of COORDINATE TRANSFORMATIONS:

Earth to Vehicle

$$\begin{vmatrix} Xc \\ Yc \\ Zc \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \begin{vmatrix} X - Xcg \\ Y - Ycg \\ Z - Zcg \end{vmatrix}$$

Vehicle to Earth

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} \begin{vmatrix} Xc \\ Yc \\ Zc \end{vmatrix} + \begin{vmatrix} Xcg \\ Ycg \\ Zcg \end{vmatrix}$$

Transformation Matrix Elements

$A_{11} = \cos\theta y \cos\theta z - \sin\theta y \sin\theta x \sin\theta z$

$A_{12} = \cos\theta y \sin\theta z + \sin\theta y \sin\theta x \cos\theta z$

$A_{13} = -\sin\theta y \cos\theta x$

$A_{21} = -\cos\theta x \sin\theta z$

$A_{22} = \cos\theta x \cos\theta z$

$A_{23} = \sin\theta x$

$A_{31} = \sin\theta y \cos\theta z + \cos\theta y \sin\theta x \sin\theta z$

$A_{32} = \sin\theta y \sin\theta z - \cos\theta y \sin\theta x \cos\theta z$

$A_{33} = \cos\theta y \cos\theta x$

EQUATIONS of STATE:

$$A_x = \dot{V}_x : A_y = \dot{V}_y : A_z = \dot{V}_z$$

$$V_x = \dot{X} : V_y = \dot{Y} : V_z = \dot{Z}$$

$$\alpha_z = \dot{\omega}_z : \alpha_x = \dot{\omega}_x : \alpha_y = \dot{\omega}_y$$

$$\omega_z = \dot{\theta}_z : \omega_x = \dot{\theta}_x : \omega_y = \dot{\theta}_y$$

STATE - SPACE FORMULATION:

$$|\dot{X}| = F_1 |X| + F_2 |U|$$

Example Equations for Each Mass

$$\begin{vmatrix} Ax \\ Ay \\ Az \\ Vx \\ Vy \\ Vz \\ \alpha z \\ \alpha x \\ \alpha y \\ \omega z \\ \omega x \\ \omega y \end{vmatrix} = F_1 \begin{vmatrix} Vx \\ Vy \\ Vz \\ X \\ Y \\ Z \\ \omega z \\ \omega x \\ \omega y \\ \theta z \\ \theta x \\ \theta y \end{vmatrix} + F_2 \begin{vmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_m \end{vmatrix}$$

NUMERICAL INTEGRATIONS:

A. Simple Derivatives Euler Method

$$X(t+\Delta t) = X(t) + \dot{X}(t)\ \Delta t$$

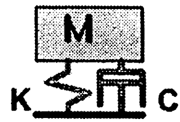
B. Linear Average Derivatives Heun Method

$$X(t+\Delta t) = X(t) + \frac{\dot{X}(t) + \dot{X}(t+\Delta t)}{2}\ \Delta t$$

C. Weighted Average Derivatives

1. Runge Kutta
2. Others

VIBRATING SYSTEMS:



Characteristic Equations

$$M \ddot{X} + C \dot{X} + K X = 0$$

$$X + \frac{C}{M} X + \frac{K}{M} X = 0$$

LaPlace Transform Form

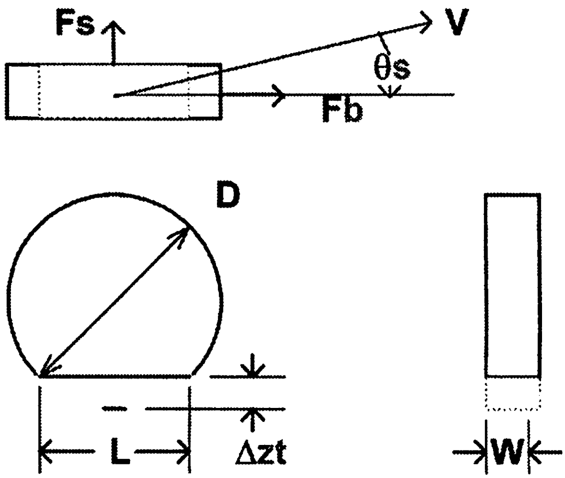
$$\left(S^2 + \frac{C}{M} S + \frac{K}{M} \right) X = 0$$

Analogy from Linear Control Theory

$$\left(S^2 + 2 \zeta \omega n S + \omega n^2 \right) X = 0$$

$$K = M (2 \pi Fn)^2$$
$$C = 2 M \zeta (2 \pi Fn)$$

TIRE CONTACT PATCH:



$$L = 2\sqrt{(D/2)^2 - (D/2 - \Delta zt)^2}$$

Normal Force

$$Fzt = P\ L\ W - C\ Vzt \quad : \quad Fzt \geq 0$$

Velocity Angle

$$\theta zv = \arctan(Vy / Vx)$$

Slip Angle

$$\theta s = \theta st + \theta zc - \theta zv$$

Slip Function

$$S = f(\theta s, Fzt) \quad \text{Empirically Derived}$$

Lateral (Side) Force

$$Fs = S\ Fzt$$

Braking Force

$$Fb = -B\ Fzt\ \mathrm{sgn}(Vxt)\ ;\ Vxt \neq 0$$

If $S^2 + B^2 < \mu g^2$, then

$$Fxc = Fb \cos \theta st - Fs \sin \theta st$$

$$Fyc = Fb \sin \theta st + Fs \cos \theta st$$

If $S^2 + B^2 \geq \mu g^2$, then

$$F^* = - \mu g \; Fzt \; \text{sgn} (Vxt)$$

$$Fxc = F^* \cos (\theta vt - \theta zc)$$

$$Fyc = F^* \sin (\theta vt - \theta zc)$$

SUSPENSION SYSTEM EQUATIONS:

Spring Forces

$$Fsp = K\ (Zt - Zc)$$

Damper Forces

$$Fsh = C\ (Vzt - Vzc)$$

Anti - Dive Link Forces

$$Fad = -Fxc\ (Zc - Zt - Hd) / Ld \quad :\text{Front}$$

or

$$Fad = Fxc\ (Zc - Zt - Hd) / Ld \quad :\text{Rear}$$

Anti - Roll Link Forces

$$Far = -Fyc\ (Zc - Zt - Hr) / Lr \quad :\text{Left}$$

or

$$Far = Fyc\ (Zc - Zt - Hr) / Lr \quad :\text{Right}$$

$$Fzc = Fsp + Fsh + Fad + Far$$

TORQUES:

$$Tzc = Fyc\ Xc - Fxc\ Yc$$

$$Txc = Fzc\ Yc + Fyc\ (Zc - Zt)$$

$$Tyc = -\ Fzc\ Xc - Fxc\ (Zc - Zt)$$

UNSPRUNG MASSES EQUATIONS:

$$Vxc = A_{11}\ Vx + A_{12}\ Vy + A_{13}\ Vz - Yc\ \omega zc$$

$$Vyc = A_{21}\ Vx + A_{22}\ Vy + A_{23}\ Vz + Xc\ \omega zc$$

$$Vzc = A_{31}\ Vx + A_{32}\ Vy + A_{33}\ Vz + Yc\ \omega xc - Xc\ \omega yc$$

$$A_{xc} = F_{xc} / M - Y_c \alpha_{zc} - X_c \omega_{zc}^2 - X_c \omega_{yc}^2$$

$$A_{yc} = F_{yc} / M + X_c \alpha_{zc} - Y_c \omega_{zc}^2 - Y_c \omega_{xc}^2$$

$$A_{zc} = (F_{zt} - F_{zc}) / M$$

$$A_x = A_{11} A_{xc} + A_{21} A_{yc} + A_{31} A_{zc}$$

$$A_y = A_{12} A_{xc} + A_{22} A_{yc} + A_{32} A_{zc}$$

$$A_z = A_{13} A_{xc} + A_{23} A_{yc} + A_{33} A_{zc} - g$$

NOMENCLATURE:

$|A| = |\dot{V}|$ = Acceleration Vector (velocity derivative)

A_n = nth element of acceleration vector

A_x, A_y, A_z = Acceleration Components in Earth Coordinates- ft/sec/sec
α_z(yaw), α_x(roll), α_y(pitch) = Angular Accelerations - radians/sec/sec
e.g. A_n = A_x, A_y, A_z, α_z, α_x, α_y

A_{ij} (i = 1,2,3 ; j = 1,2,3) = Coordinate Transformation Matrix Elements

B = Braking Function at Tire Contact Patch - dimensionless

C = Damper Constant - pound*second/foot

D = Diameter of Tire - inches

F_{ad}, F_{ar} = Forces at Suspension Anti-Dive and Anti-Roll Elements - pounds

F_n = Natural Frequency - hertz (cycles/second)

F_s, F_b, F_{zt} = Forces at Tire Contact Patch - slip, braking, normal - pounds

F_{sp}, F_{sh} = Forces at Suspension Spring and Shock Absorber - pounds

F_x, F_y, F_z = Force Components in Earth Coordinates - pounds

F_{xc}, F_{yc}, F_{zc} = Force Components in Vehicle Components - pounds

F_1, F_2 = Expressible Mathematical Functions

g = Gravity Constant = 32.174 feet/second/second

H_d, H_r = Constants Determined by Suspension Geometry

I_{zc}, I_{xc}, I_{yc} = Moments of Inertia about Vehicle Axes - pound*ft*sec*sec

K = Spring Constant - pounds/foot

L = Length of Tire Contact Patch - inches

L_d, L_r = Constants Determined by Suspension Geometry

M = Mass - pound*second*second/foot

μg = Coefficient of Friction at Tire Contact Patch - dimensionless (decreasing function of speed)

P = Air Pressure of Tire - pounds/inch/inch

π = 3.141593 to seven significant figures

S = Slip Function at Tire Contact Patch - dimensionless

t = Time - seconds

Δt = Time Step for Numerical Integration - second

T_{zc}(yaw), T_{xc}(roll), T_{yc}(pitch) = Torques about Vehicle Axes - feet*pounds

θ_s = Slip Angle at Tire Contact Patch - radians

θ_{st} = Steering Angle of Tire in Vehicle Coordinates - radians

θ_{zc} = Yaw Angle of Tire in Earth Coordinates - radians

θ_{zv} = Velocity Angle at Tire in Earth Coordinates - radians

|U| = Control (external input) Vector

U_m = mth element of control vector

$|V| = |\dot{X}|$ = Velocity Vector (position derivative)

V_n = nth element of velocity vector

V_x, V_y, V_z = Velocity Components in Earth Coordinates - feet/second

ω_z(yaw), ω_x(roll), ω_y(pitch) = Angular Velocities - radians/second

e.g. V_n = V_x, V_y, V_z, ωz, ωx, ωy

V_{xc}, V_{yc}, V_{zc} = Velocity Components in Vehicle coordinates - ft/sec

V_{zt} = Normal Velocity at Tire Contact Patch - feet/second

W = Width of Tire Contact Patch - inches

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ζ = Damping Ratio - decimal fraction of critical value

|X| = State (position) Vector

Xn = nth element of state vector

X, Y, Z = Position Coordinates in Earth Coordinates - feet

θz (yaw), θx (roll), θy (pitch) = Angular Coordinates - radians

e.g. Xn = X, Y, Z, θz, θx, θy, Vx, Vy, Vz, ωz, ωx, ωy

Xc, Yc, Zc = Position Coordinates in Vehicle Coordinates - feet

Xcg, Ycg, Zcg = Center of Mass Position in Earth Coordinates - feet

Δzt = Compression at Tire Contact Patch - inches

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