Journal of the National Academy of Forensic Engineers®

NATIONAL ACADEMY OF FORENSIC ENGINEERS

http://www.nafe.org
ISSN: 2379-3252

Vol. XXVI No. 2 December 2009

Forensic Engineering Analysis of Head Impact From Falling Picture Frame

By Michael Kravitz, P.E. (451F)

Abstract

This paper will examine the effect of an oak picture frame that was wall mounted around a flat screen television. The Frame fell from the flat screen television down 6 inches onto a fireplace mantel, and then rotated away from the wall, and the upper edge of the frame struck the Plaintiff on the head. The Plaintiff was sitting on an ottoman in front of the fireplace facing away from the fireplace and was not aware of the impending impact. Plaintiff sued the premises' owner, claiming various head, neck and dental injuries. An engineering analysis was performed to assess the magnitude of the force applied to Plaintiff's neck with an analytical model which was developed to check the analysis claimed by the opposing Expert Biomechanical Engineer. The writer, not a Biomechanical Engineer, analyzed the magnitude of the compressive neck loading. The Writer developed a head-neck model that was utilized in the analysis. The neck was modeled as a spring; the head a sphere; the neck-spring was mounted on a rigid platform, i.e., the shoulder. Based on research, ranges of spring constants were estimated for the neck. The Writer settled on a maximum and minimum neck spring constant with correlating damping. The issue in the case for the Writer was the magnitude of the force applied to the head and neck and whether that force was sufficient to cause the injuries claimed by the Plaintiff. The Bio-Mechanical Engineering issues are addressed in the corresponding paper by Jon O. Jacobson, Ph.D., P.E. The case went through many pre-trial discussions and scheduling, along with the deposition of the Writers which were taken by Plaintiff's Attorney. A formal written report was not prepared. Mathcad calculations with sidebar explanations were prepared and supplemented with an oral report to the clients. As a result of the analysis, the case settled at the eleventh hour prior to trial.

Keywords

Neck, Head, Moment Of Inertia, Spring Constant, Momentum, Differential Equation NATO, Injury.

Events Leading To Trial

The setting was a hotel suite where a holiday party was taking place. Groups of people were sitting at various areas of the suite. The group in question was sitting in front of the fireplace where there was a sofa facing the fireplace. The Plaintiff was seated on an ottoman facing opposite the fireplace. Above the fireplace was a flat screen television mounted to a false plywood wall. An oak frame, four inches (4") in width around weighing 21 pounds, surrounded the television and was mounted on the wall using slotted aluminum straps that were screwed into the back of the oak frame. The aluminum straps fitted over

Michael Kravitz, P.E., Consulting Engineer, 484 West 43rd Street, Suite 32S, New York, NY 10036

screw heads secured to the false plywood wall. Apparently, the oak frame was dislodged from its original secured position and merely rested on the flat screen television and therefore, prior to the incident, the oak frame was not securely mounted with the screws and aluminum straps. Because the free space between the frame and television was small, it was not distinguishable that the oak frame was resting or draped on the television. A youngster was playing with toys on the fireplace mantel surface and allegedly bumped the oak frame which caused it to fall from its rest or draped position on top of the television screen. The frame dropped vertically down onto the fireplace mantel then rotated approximately ninety degrees where the upper portion of the frame struck the Plaintiff on top of the head. Plaintiff incurred a small cut on the top of the head but did not lose consciousness nor did she feel any dizziness. An ice bag was placed on plaintiff's head and the plaintiff continued on with the gathering. It is noteworthy that if the oak frame was mounted properly, that is, secured with the aluminum strap and screw, to dislodge the frame from the screw/slot configuration would have required a upward force of 42 pounds as measured with a Chatillon force gauge, or approximately twice the weight of the frame.

The Writer was called approximately two years after the event to examine the frame, frame mounting, and take measurements and photographs. The Writer performed an engineering analysis to determine the force the frame delivered to Plaintiff's head, and suggested to his clients to retain a Biomechanical Engineer to further counter Plaintiff's experts. The parties could not come to a settlement, so the case moved along toward trial.

Case Analysis

The Writer generated a model of the head and neck using a sphere and spring along with a three dimensional sketch of the frame. By using 3-D modeling software, a simulation of the motion of the frame falling from the wall downward to the mantel and then rotating to impact the top of the head was developed. This was created so that all concerned would understand the motion of the frame as it fell and then rotated to strike Plaintiff's on top of the head.

Research was conducted to obtain a value(s) to be used for treatment of the neck as a spring in compressive loading. Maximum and minimum values were selected for the neck response as a linear spring with coulomb damping (Maxwell element). The minimum value used was obtained from the paper titled "Neuromuscular control and attention level in cervical spine dynamics: Experimental study and multibody model computer simulation"[1] and was estimated to be 30kN/m with damping of 400Ns/m. The maximum value was obtained from the paper titled, "Dynamic Characteristics of the Intact, Fused, and Prosthetic-Replaced Cervical Disk"[2] estimated at 425kN/m with a damping of 1500Ns/m.

For visualization, below are frames from the demonstration model of the frame falling onto a mantel, then rotating and striking a "head" on a "neck". It was estimated the top of the injured party's head was the same elevation as the mantel onto which the frame dropped, and the frame, when it rotated and struck the individual, rotated approximately 90 degrees as illustrated in figure 5 and figure 7.

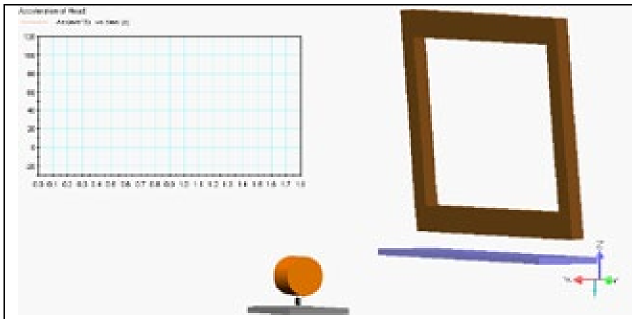


Fig. 1

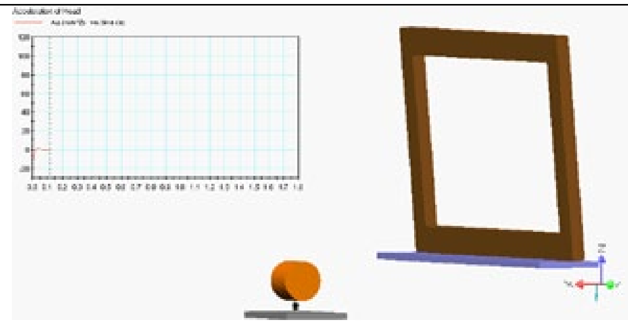


Fig. 2

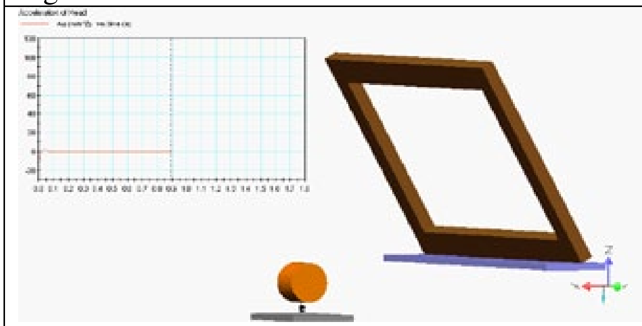


Fig. 3

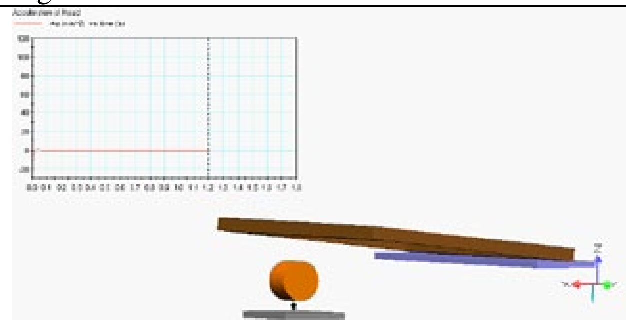


Fig. 4

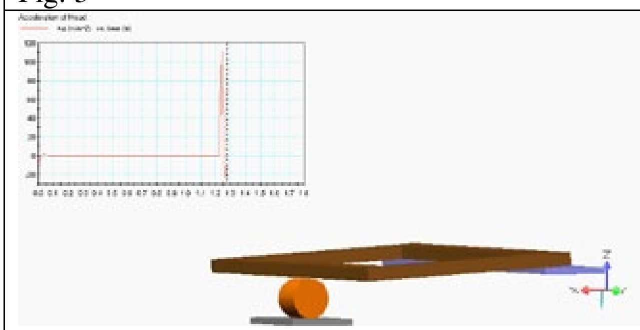


Fig. 5

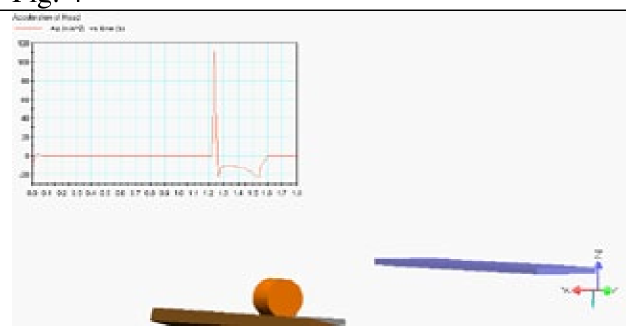


Fig. 6

Calculations

The first item was to calculate the velocity that the frame strikes the head. Using energy, the Potential Energy of the frame was equated to its Kinetic Energy plus its Rotational Energy. In order to calculate the frame-head impact the calculation of the mass moment of inertia of the frame was necessary. The calculation of the mass moment of inertia (I_{cc}) of the frame was performed by subtracting the inner opening from the outer perimeter of the frame. The rotation would be about the bottom of the frame. I_{cc}, by applying the parallel axis theorem would be:

$$I_{cc} = \frac{M_{total}}{3} \cdot \left[h^2 - \frac{(L - 2 \cdot b_f) \cdot (h - 2 \cdot b_f)}{L \cdot h} \cdot \left(h^2 - h \cdot b_f + b_f^{\,2} \right) \right]$$

$$I_{cc} = 2.11 \, ft \cdot lb \cdot s^2$$

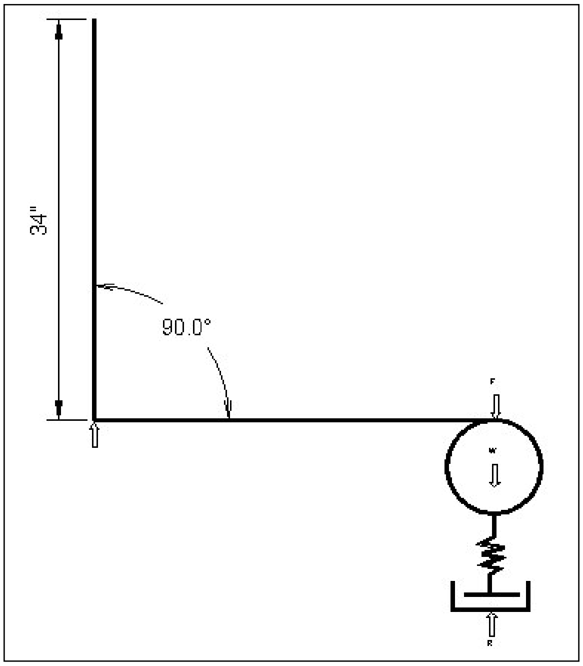


Figure 7
The free body diagram of the frame striking the head where the neck acts like a spring with damping.

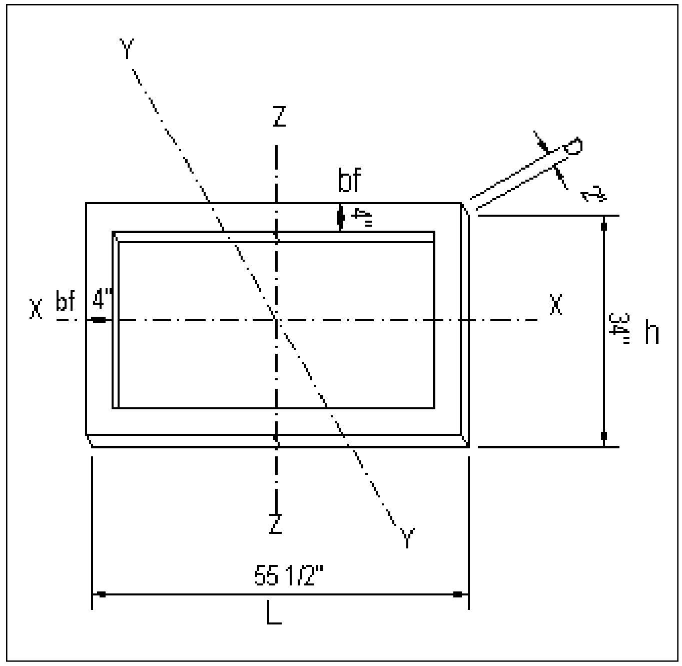


Figure 8
Diagram of the frame that strikes head along with the frame dimensions.

The calculation of the angular velocity ω of the falling frame was obtained by equating the Potential Energy to the Linear Kinetic Energy and the Angular Kinetic Energy in the general expression.

(**w** x **h/2**) = 1/2 · **w/g** · $\mathbf{v}^2$+ 1/2 · $\mathbf{I_{cc}}$ · ω^2. Where **h** is the vertical height of the frame, **w** is the weight of the frame, **v** and r are the linear and angular velocity of the frame respectively taken about the axis of rotation, the bottom of the frame, $\mathbf{I_{cc}}$ is the mass moment of Inertia of the frame about the bottom of the frame, $\mathbf{b_f}$ is the frame width around the opening, and **g** is the acceleration due to gravity.

$$w \cdot \frac{h}{2} = \frac{1}{2} \cdot \frac{w}{g} \cdot \left(\omega \cdot \frac{h}{2} \right)^2 + \frac{1}{2} \cdot I_{cc} \cdot \omega^2$$

$$\omega = \sqrt{\frac{12 \cdot g \cdot h}{\left[3h^2 + \frac{4 Area_{total}}{\left(Area_{total} - Area_{in} \right)} \cdot \left[h^2 - \frac{\left(L - 2 \cdot b_f \right) \cdot \left(h - 2 \cdot b_f \right)}{L \cdot h} \cdot \left(h^2 - h \cdot b_f + b_f^2 \right) \right] \right]}}$$

$$\omega = 4.18 \frac{1}{s}$$

ω is the angular velocity of the frame after rotating 90 degrees.

It is important to note that Plaintiff's Expert treated the frame as a two dimensional solid rod or cylinder and calculated the velocity of the end of the cylinder using the equation for mass moment of inertia from the book written by Arthur C. Damask and Jay N. Damask as:

$$v = \sqrt{3 \cdot g \cdot h}$$

v = 16.5 fps

(From Injury Causation Analysis: Case Studies and Data Sources by Arthur C. Damask and Jay. N. Damask, The Michie Company, Law Publishers, Charlottesville, Virginia, Appendix P, Speed at which a standing human falls.

A velocity of 16.50 fps was Plaintiff's Expert calculated velocity of the frame striking the head using a two dimensional cylinder, versus the frame velocity using the appropriate mass moment of inertia of the frame which resulted in the velocity of:

v = 11.2 fps.

The momentum exchange from the frame to the head produces a velocity of the head on the neck. The velocity of head on the neck is

$$v_2 := v \cdot \frac{w}{w + Wt}$$

V_2 = 7.6 fps

Following are the data for calculations:

Wt. = weight of head = 9.8 lbs

W= weight of frame = 21.2 lbs

L = length of frame = 55-1/2 inches

h = height of frame = 34 inches

bf = Frame width = 4 inches

k = Spring Constants for the neck = 30000 N/m to 425000 N/m (172 lb/in to 2427 lb/in)

b = Damping constant for the neck = 400 N/m/s to 1500 N/m/s (3 lb/in/s to 38 lb/in/s)

With the velocity of the head on the neck resolved, the calculation to determine the neck force can proceed. The acceleration of the neck due to the acceleration of the head on the neck will determine the force the head exerts on the neck. Using Newton's 2nd Law of Motion, F=Ma, and a free body diagram, the equation of dynamic motion can be established. (See appendix for the detailed differential equation solution for the low and high estimate of the spring constant and damping coefficient used for the neck.)

$y(t) := \frac{v_2}{\kappa} \cdot e^{-\gamma \cdot t} \cdot \sin(\kappa \cdot t)$	$y'(t) := \left(\frac{d}{dt} y(t)\right)$	$y''(t) := \left(\frac{d^2}{dt^2} y(t)\right)$
Displacement	Velocity	Acceleration

Using the maximum and minimum spring constants found for the neck of 425kN/m and a low value of 30kN/m along with a damping of 1500Ns/m and 400Ns/m respectively, the graphs below were generated for acceleration and force versus time base on the solved differential equations.

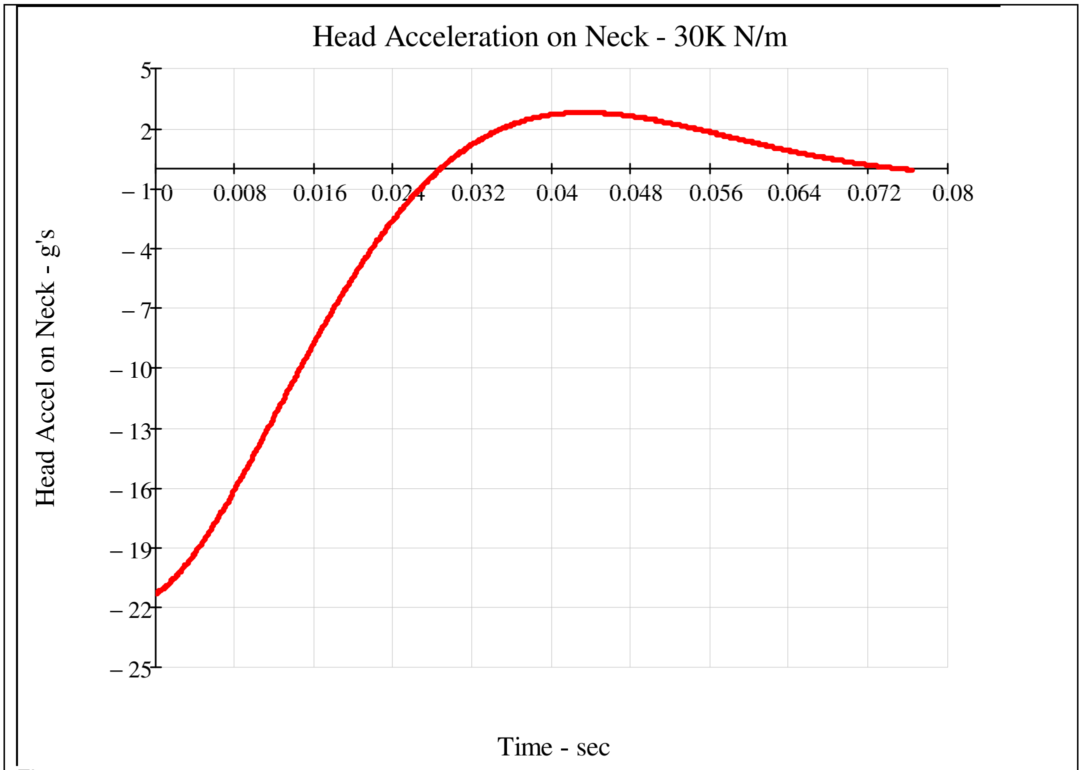


Fig. 9

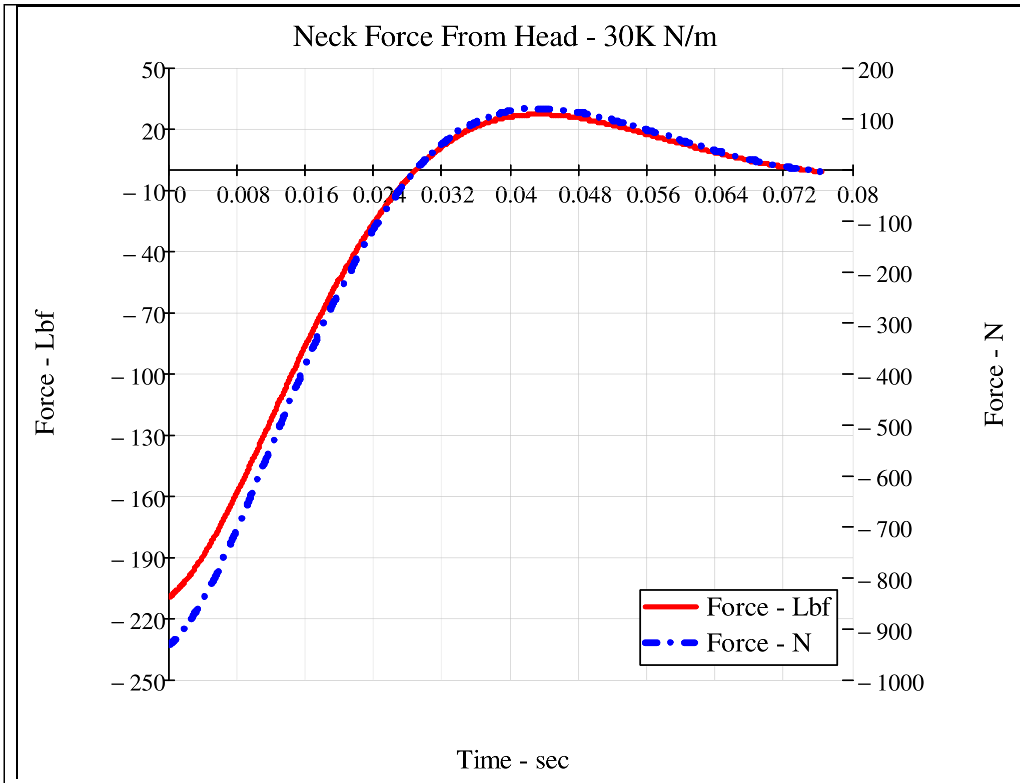

Neck Force From Head - 30K N/m
Force - Lbf
Force - N
Time - sec
50
20
– 10
– 40
– 70
– 100
– 130
– 160
– 190
– 220
– 250
200
100
– 100
– 200
– 300
– 400
– 500
– 600
– 700
– 800
– 900
– 1000
0
0.008
0.016
0.024
0.032
0.04
0.048
0.056
0.064
0.072
0.08
Force - Lbf
Force - N

Fig. 10

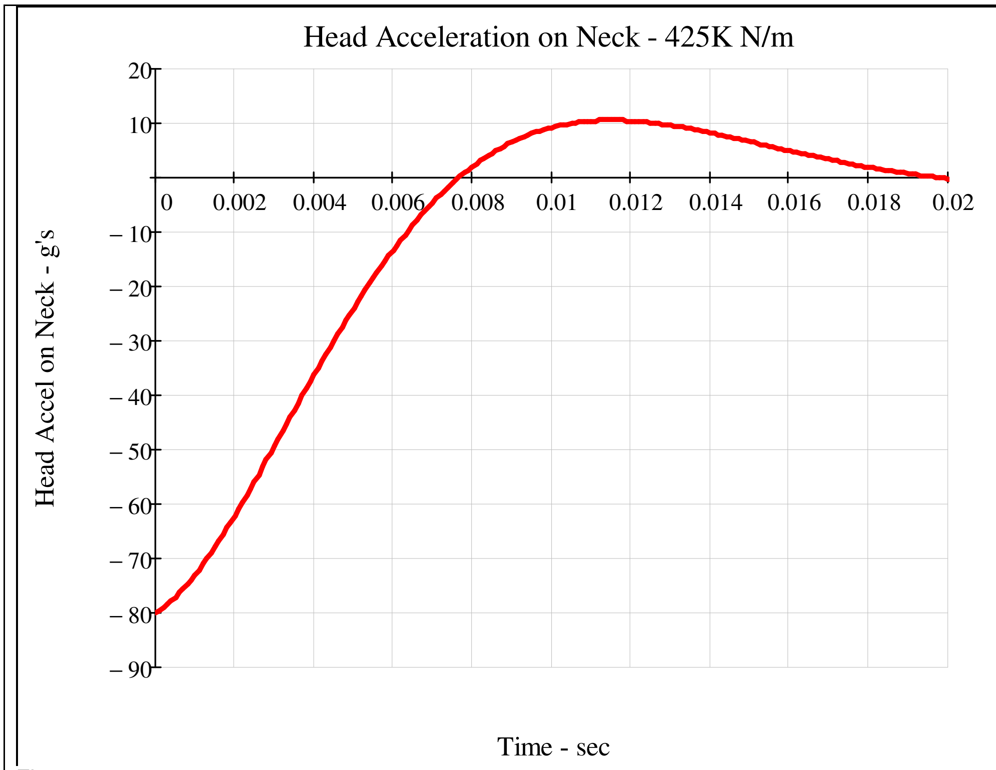

Head Acceleration on Neck - 425K N/m
Head Accel on Neck - g's
Time - sec
20
10
– 10
– 20
– 30
– 40
– 50
– 60
– 70
– 80
– 90
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016
0.018
0.02

Fig. 11

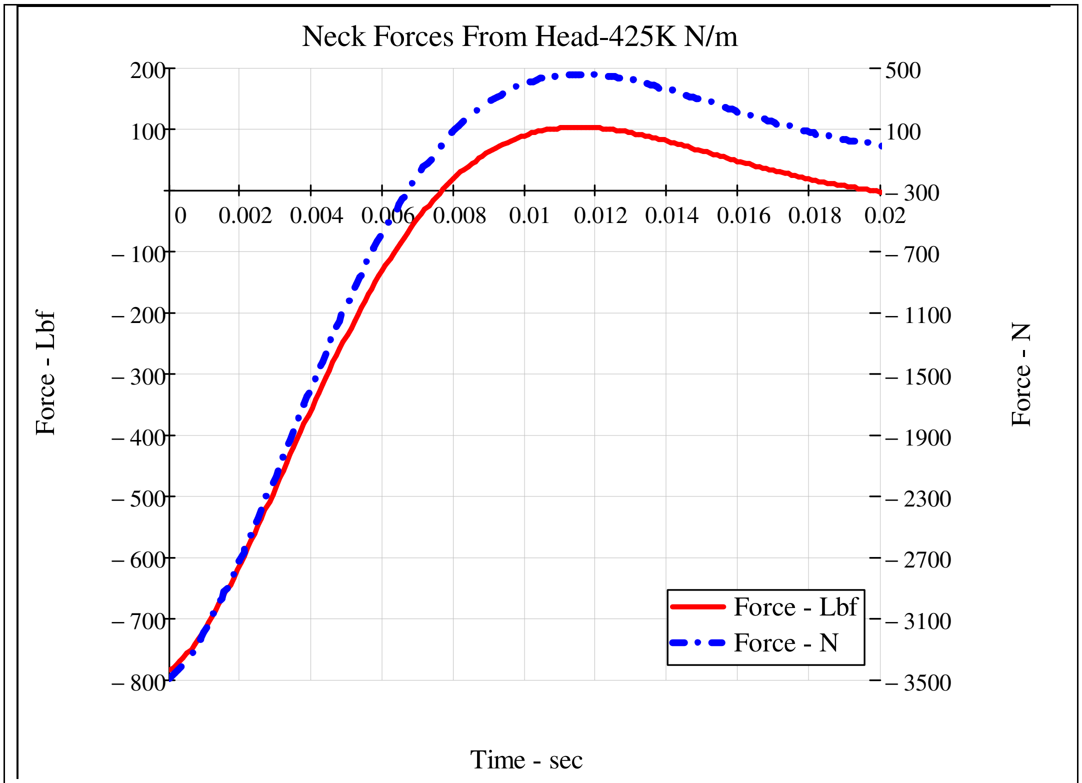


Fig. 12

The plots of the stiff spring and soft spring constants return equal impulses. The plot for the "soft" spring constant of 30kN/m and damping of 400Ns/m produces the values of impulse, delta-v and acceleration:

	$t_1 := 0\text{s}$	$t_2 := 0.028\text{s}$
$\text{Impulse} := \int_{t_1}^{t_2} \text{force}(t)\, dt$	$\text{Impulse} = -2.96\,\text{lb}\cdot\text{s}$	
$\text{deltaV} := \dfrac{\text{Impulseg}}{\text{Wt}}$	$\text{deltaV} = -9.7\,\dfrac{\text{ft}}{\text{s}}$	$\dfrac{y''(0)}{g} = -21.3$

The plot for the "stiff" spring constant of 425kN/m and damping of 1500Ns/m produces the values of the values of impulse, delta-v and acceleration:

	$t_1 := 0s$	$t_2 := 0.0074s$
$Impulse := \int_{t_1}^{t_2} force(t)\,dt$	$Impulse = -2.96\,lb\cdot s$	
$deltaV := \frac{Impulse}{Wt}$	$deltaV = -9.7\frac{ft}{s}$	$\frac{y''(0)}{g} = -80$

The differences are the impulse time values although the impulses are equal.

Further research of the NATO Research and Technology Organization (RTO), TR-HFM090-03.pdf, Chapter 3 – Injury Criteria and Tolerance Levels based on this study, Mertz [Mertz, 1978][3] derived two injury tolerance curves based on the upper neck axial compression force measured on a 50th percentile male Hybrid III (see Figure 3.16). The coordinates of the "Upper" curve are 0 ms and 6670 N, 35 ms and 1110 N, and greater than 35 ms, 1110 N.

The coordinates of the "Lower" curve are 0 ms and 4000 N, 30 ms and 1110 N, and greater than 30 ms, 1110 N. To evaluate the neck load signal, pairs of points (force, duration) are plotted on the graph (shown in Figure 3.16) with the two injury assessment curves. The points are connected together by a series of straight lines. If any of the line segments lie above the upper curve (red), the neck axial compression force is considered to have the potential to produce serious neck injury. If any of the line segments lie above the lower curve (blue), but still below the upper (red) curve, the potential for neck injury from the axial compressive force is considered less likely.

If the lines fall below the lower (blue) curve, the probability of neck injury from axial compressive force is considered remote.

These levels were proposed for an adult population that was considerably older (exact age range not known) and much less conditioned than a high school football athlete. The time durations were determined from the loading times observed during the experiment, which were on the order of 30 – 40 ms. The full study is referenced and can be obtained via their website.

NATO OTAN

INJURY CRITERIA AND TOLERANCE LEVELS

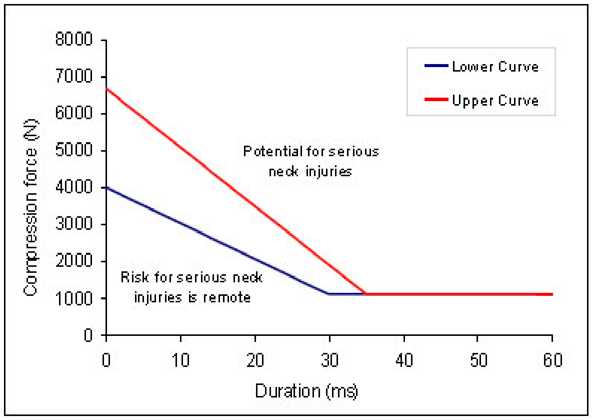


Figure 3.16: Injury Tolerance Curves for Axial Neck Compression Force when Using a Hybrid III 50th Percentile Male ATD [Mertz, 1978].

The NATO Research and Technology Organization (RTO), TR-HFM-090-03.pdf, Chapter 3 – Injury Criteria and Tolerance Levels. Based on this study, Mertz [Mertz, 1978] derived two injury tolerance curves based on the upper neck axial compression force measured on a 50th percentile male Hybrid III (see Figure 3.16).

Fig. 13

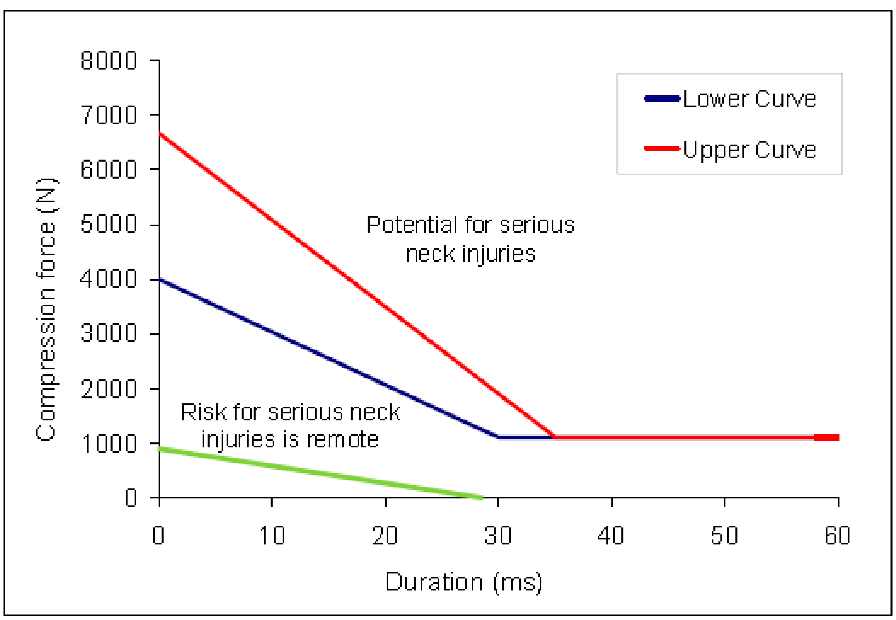


Figure G.2: Injury Tolerance Curves for Axial Neck Compression Force when Using a Hybrid III 50th Percentile Male ATD [Mertz, 1978].

The lowest line is the Injury Tolerance Curve for the soft spring constant 30kN/m.

Fig. 14

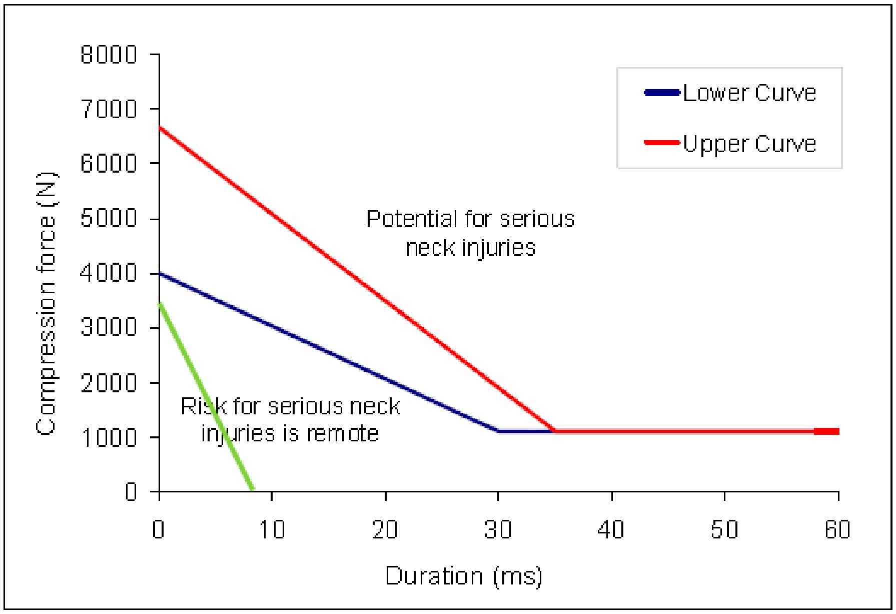


Figure G.2: Injury Tolerance Curves for Axial Neck Compression Force when Using a Hybrid III 50th Percentile Male ATD [Mertz, 1978].

The lowest line is the Injury Tolerance Curve for the soft spring constant 425kN/m.

Fig. 15

Using the Injury Tolerance Curves for Axial Neck Compression Force for both the neck spring constant of 30kN/m and 425kN/m shows that the calculated lines (green line or lowest line) fall under the "Risk for serious neck injury is remote". (See Figures 14 and 15).

During deposition, the Writer was asked several times if he was a Biomechanical Engineer, and the response was no, that he was determining the loading on the neck which can be performed by most engineers using dynamic loading principals and Newton's Laws of Motion. It again should be noted that the opposing expert Biomechanical Engineer calculated and opined that only half of the weight of the frame was applied to the head. The reasoning for this was that the frame was statically or simply supported when one end of the frame struck the head; the other end of the frame was supported by the mantel. This was not correct because the frame was in rotation and the head felt the entire weight of the frame. (See calculations and analysis in the Appendix.) One can only speculate that the opposing Expert Biomechanical Engineer's reason for treating the frame as a simply supported beam was that the force on the head which he calculated was unreasonably large, and that the force had to be reduced to a value that was more consistent with the injuries claimed. The opposing Expert Biomechanical Engineer's analysis was the following:

Delta-V = 16.5 fps from the calculation of the falling cylinder.

Weight of frame = 21 lb

Delta-t = 2.8 ms (As an aside, the writer could not identify the source of this value in the published data supplied by Plaintiff's Biomechanical Engineer. As a matter of fact, using these values, with a momentum exchange, the velocity of the head on the neck results in a velocity of head on the neck of 11.2 fps and a force of 307 g's. The corresponding spring constant for the neck would calculate to approximately 3,000 kN/m and damping of approximately 4,000 Ns/m, which is not realistic.)

Force = W/g * delta-v / delta-t; = 21 lb / 32.2 ft/s2 * 16.5 ft/s / 0.0028s

Force = 3,843 lbf (17,095 N)

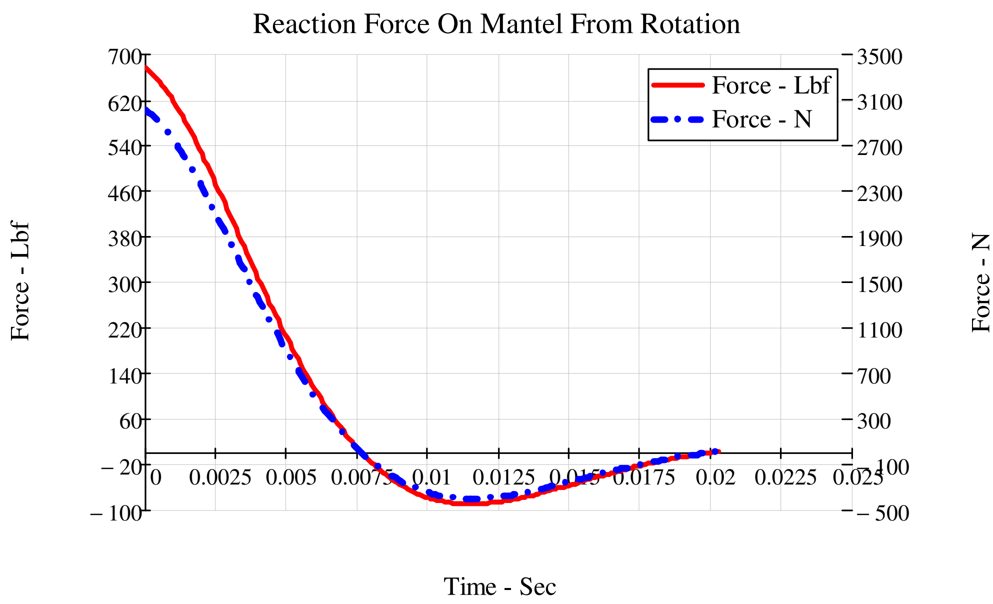
The magnitude of this force was greater than the magnitude of the force which caused the injuries; the opposing Expert Biomechanical Engineer reduced the calculated value by 50% reasoning that only half the weight of the frame contacted the head or 1,922 lbf or 8,549 Newtons. This value would change the value from "unreasonable" and place the Plaintiff's injury in the "Potential for Serious Neck Injury" category if one chose to use the Mertz, 1978 study which derived the two injury tolerance curves.

The results of whether the frame rose up from the mantel to cause the head to feel the full weight of the frame were calculated using the neck spring constant of 425kN/m and the damping of 1500Ns/m at time = 0. The neck reaction is maximum at (-3488) Newtons upward resisting motion, and the mantel reaction is approximately (+3010) Newtons downward resisting motion. The spring resists force, therefore, the head feels the full weight of the frame at impact because the mantel end of the frame was up, and the calculated force was resisting motion in the downward direction. (See the calculations in the appendix.)

The calculations above determine the direction and magnitude of forces on the mantel end of the frame as it strikes the Plaintiff. Because springs resist motion, the positive force on the mantel end of the frame shows that it is resisting the upward motion of that end of the frame and that the Plaintiff felt the full weight of the frame. Therefore, Plaintiff's Expert was not correct in opining that half of the frame weight is what struck the Plaintiff. (See Fig. 16).

Conclusion

Using dynamic mechanics, with a reasonable model, the range of forces on the neck can be calculated, and this information can be relayed to the Biomechanical Engineer and/or medical experts to perform further analysis on the level of injury sustained. In addition, the impulse was the same for the varied values of the neck spring constant because the velocity of the impact governs. The stiffer neck spring constant generates a higher g-force because the impulse time is short; a softer neck spring constant generates a lesser g-force because the impulse time is longer.



At time = 0, the bottom edge of the frame (which has been resting on the mantel as the frame rotates down onto her head) will want to "jump" upward off the mantel at the moment of head impact. Graph of the force acting on the mantel end of the frame. Positive force is a downward resisting the upward motion of the end of the frame. Therefore, the Plaintiff feels the entire weight of the frame not half the weight as Plaintiff's expert opined.

Fig. 16

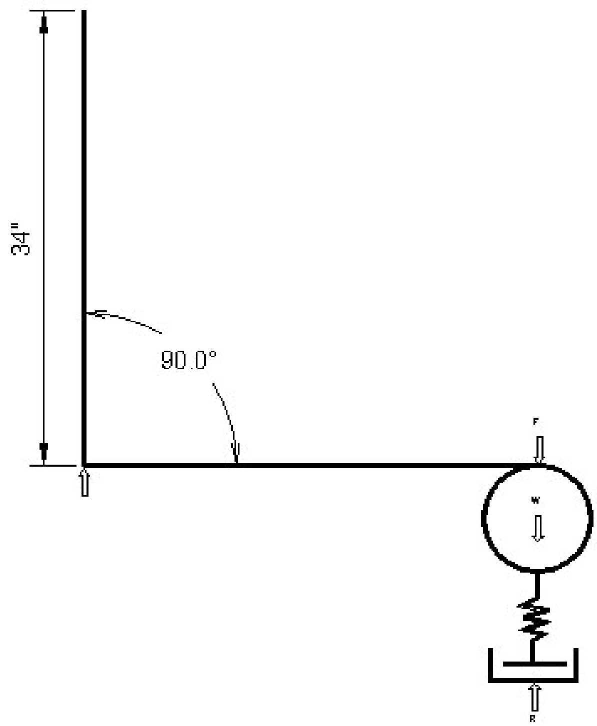
References

1. "Neuromuscular control and attention level in cervical spine dynamics: Experimental study and multibody model computer simulation", by F. Zuppichini1, P.B. Pascolo2, G. Antonutto3, 1 Italian Society for Biomechanics of Traumatic Lesions, Verona, Italy, 2 Bioengineering Dept., International Center for Mechanical Sciences, Udine, Italy, 3 Dept. of Sciences and Biomedical Technologies, University of Udine, Italy.

2. "Dynamic Characteristics of the Intact, Fused, and Prosthetic-Replaced Cervical Disk" by Michael C. Dahl1**,** Applied Biomechanics Laboratory, Department of Mechanical Engineering, University of Washington, Seattle, WA 98109; Jeffrey P. Rouleau, Cervical/Trauma Division, Medtronic Sofamor Danek, Inc., 710 Medtronic Parkway, Minneapolis, MN 55432 Stephen Papadopoulos, Barrow Neurosurgical Associates, 2910 N. 3rd Ave., Phoenix, AZ 85013; David J. Nuckley, Randal P. Ching, Applied Biomechanics Laboratory, Department of Mechanical Engineering, University of Washington, Seattle, WA 98109

3. RTO-TR-HFM-090 Test Methodology for Protection of Vehicle Occupants against Anti-Vehicular Landmine Effects.

http://www.rta.nato.int/pubs/rdp.asp?RDP=RTO-TR-HFM-090



Frame rotates and falls on head.

The free body diagram of the frame striking the head where the neck acts like a spring with damping.

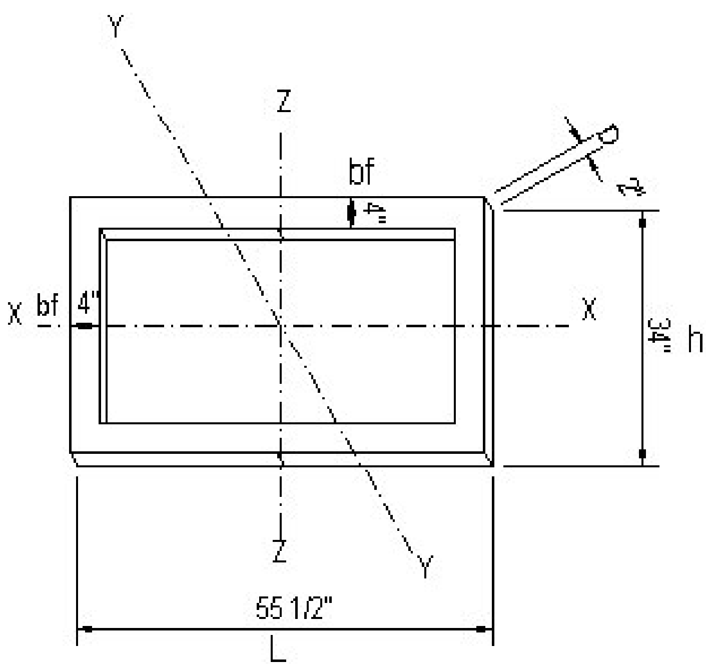


Diagram of the frame that strikes head.

Frame dimensions.

Picture frame falls on mantel, rotates and strikes the head of woman sitting on an ottaman. The frame falls directly down (vertically) strikes a fireplace mantel and rotates approximatley 90 degrees to strike woman on top of her head.

Wt= weight of head = 9.8 lbs
w = weight of frame = 21.2 lbs
L = length of frame = 55-1/2 inches
h = height of frame = 34 inches
b = Frame width = 4 inches
k = Spring Constants for the neck = 30000 N/m
b = Damping for the neck = 400 N/m/s

Potential Energy = Kinetic Energy + Rotational Energy
$w \times h/2 = 1/2 \times w/g \times \omega^2 \times (h/2)^2 + 1/2 \times I_{cc} \times \omega^2$

$L := 55.5in \quad Wt := 9.8lb \quad w := 21.2lb \quad h := 34in \quad b_f := 4in$

$Area_{total} := L \cdot h \quad Area_{in} := (L - 2b_f)\cdot(h - 2\cdot b_f) \quad w_{total} := w\cdot\left(\frac{Area_{total}}{Area_{total} - Area_{in}}\right)$

$k := \frac{30000\frac{N}{m}}{g} \quad k = 2056\frac{lb}{ft}$ Spring stiffness of neck per reference note (1).

$M := \frac{Wt}{g} \quad \omega_o := \sqrt{\frac{k}{M}} \quad \omega_o = 82.2\frac{1}{s}$ Natural frequency ω_o of the head.

$$\omega := \sqrt{\frac{12\cdot g\cdot h}{\left[3h^2 + \frac{4Area_{total}}{(Area_{total} - Area_{in})}\cdot\left[h^2 - \frac{(L - 2\cdot b_f)\cdot(h - 2\cdot b_f)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]}}$$

$\omega = 4.18\frac{1}{s}$ Angular velocity of frame as it rotates through $\pi/2$.

$v := \omega\cdot\left(h - \frac{b_f}{2}\right) \quad v = 11.2\frac{ft}{s}$ Velocity that frame strikes head.

$v_2 := v\cdot\frac{w}{w + Wt} \quad v_2 = 7.6\frac{ft}{s}$ Velocity of head after impact using Momentum analysis.

$b := \frac{400}{g}\frac{N \cdot s}{m}$ $b = 27.4\frac{s \cdot lb}{ft}$ Damping constant of neck per.

$\gamma := \frac{b}{2 \cdot M}$ $\gamma = 45\frac{1}{s}$ $T := \frac{2 \cdot \pi}{\omega_o}$ $T = 0.08\,s$

$f := \frac{1}{T}$ $f = 13.1 \cdot Hz$

$\kappa := \sqrt{{\omega_o}^2 - \gamma^2}$ $\kappa = 68.7\frac{1}{s}$ $t := 0ms, 0.1ms..T$ The contact time in milliseconds as the frame strikes the head to determine the force in g's.

Mathematical differential equation solution of head on spring being struck with force and compressing the spring (neck).

$y(t) := \frac{v_2}{\kappa} \cdot e^{-\gamma \cdot t} \cdot \sin(\kappa \cdot t)$ $y'(t) := \left(\frac{d}{dt}y(t)\right)$ $y''(t) := \left(\frac{d^2}{dt^2}y(t)\right)$

$y(t) =$ (mm)

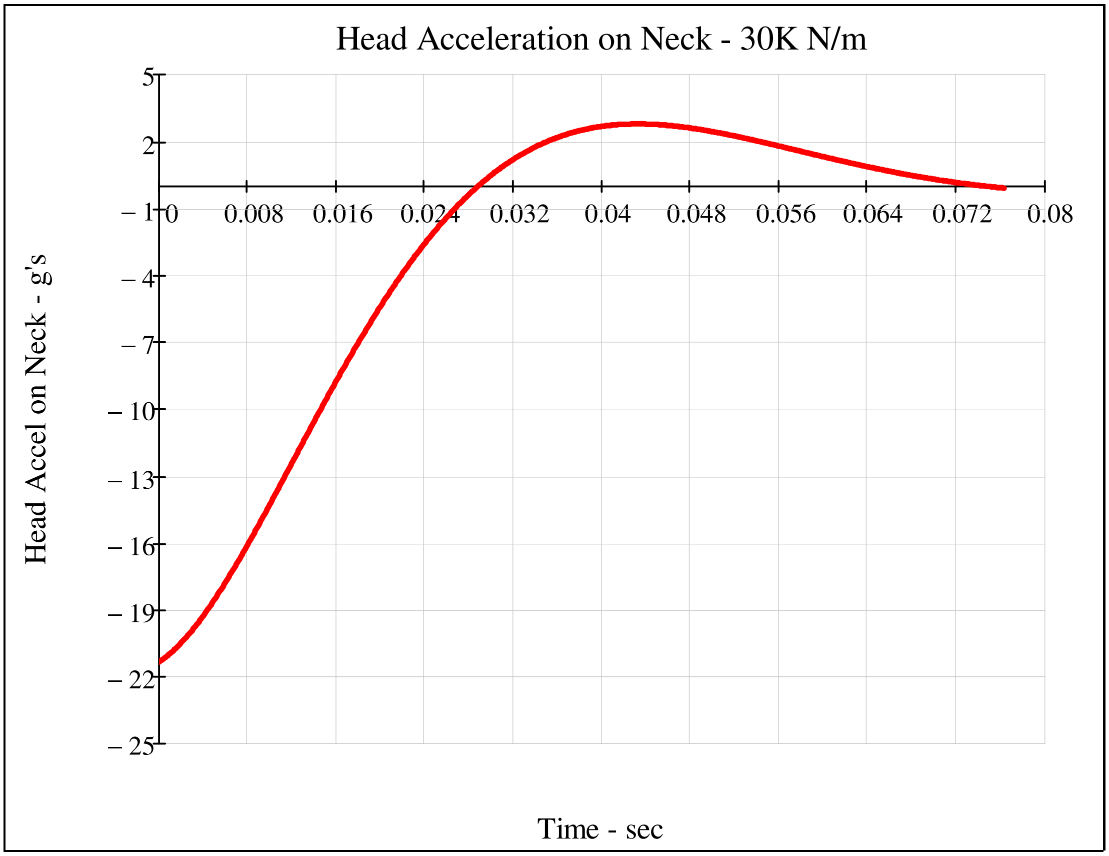
0
0.23
0.46
0.69
0.91
1.14
...

$y'(t) =$ ($\frac{ft}{s}$)

7.63
7.56
7.49
7.42
7.36
7.29
...

$y''(t) =$ ($\cdot g$)

-21.3
-21.3
-21.3
-21.2
-21.2
-21.2
...


Head Acceleration on Neck - 30K N/m
Head Accel on Neck - g's
Time - sec
5
2
– 1
– 4
– 7
– 10
– 13
– 16
– 19
– 22
– 25
0
0.008
0.016
0.024
0.032
0.04
0.048
0.056
0.064
0.072
0.08

$$\text{force}(t) := \frac{Wt}{g} \cdot y''(t)$$

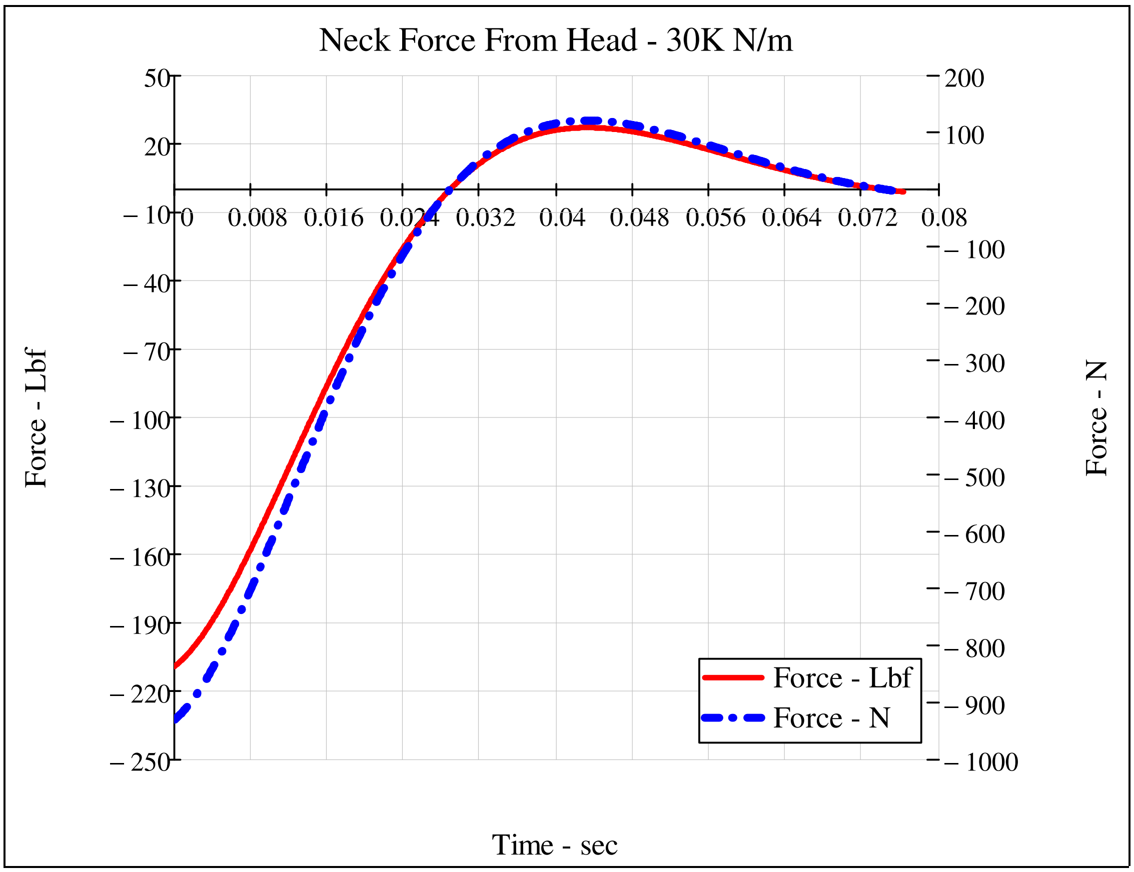
The force that the neck feels due to the acceleration of the head using Newton's 2nd Law; Force = Mass x Acceletation
F = M x a
Where acceleration = y''(t)

$\text{force}(t) =$

-209	lb
-209	
-208	
-208	
...	

$\text{force}(t) \cdot g =$

-930	·N
-929	
-927	
-926	
...	



The Green line represents the Impulse of the frame/head strike on the neck as calculated.

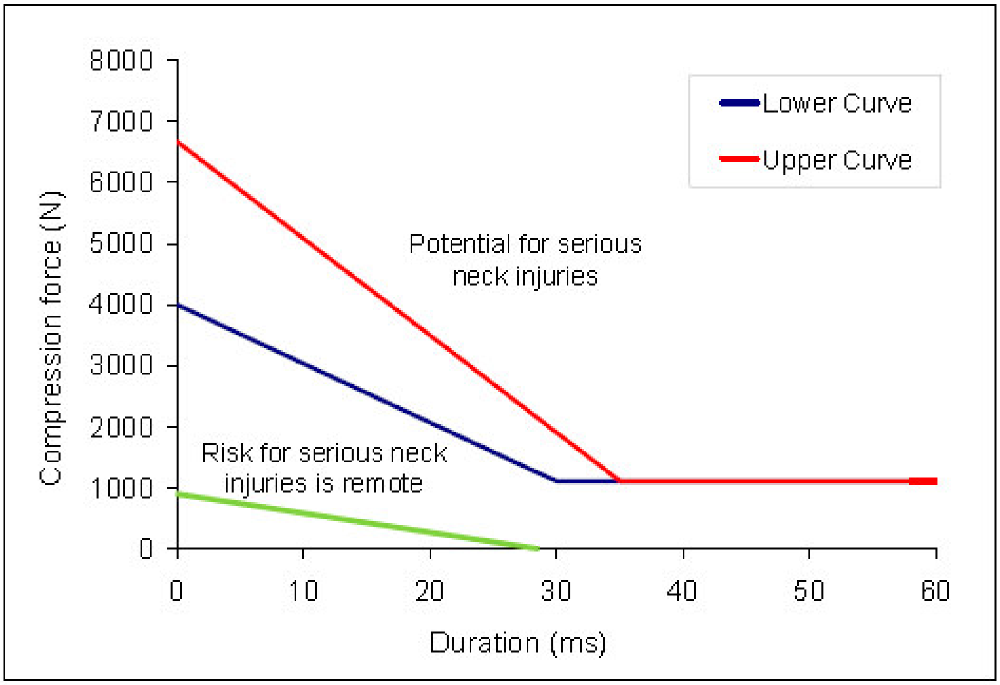


Figure G.2: Injury Tolerance Curves for Axial Neck Compression Force when Using a Hybrid III 50th Percentile Male ATD [Mertz, 1978].

The following analysis is to calculate the impulse, delta-V and HIC value that the neck feel as a result of the frame inpact.

$t_1 := 0s$ $\quad$ $t_2 := 0.028s$

$$\text{Impulse} := \int_{t_1}^{t_2} \text{force}(t)\,dt \qquad \text{Impulse} = -2.96\,s\cdot lb$$

$$\text{deltaV} := \frac{\text{Impulse}\cdot g}{Wt} \qquad \text{deltaV} = -9.7\,\frac{ft}{s} \qquad \frac{y''(0)}{g} = -21.3$$

This analysis is to determine if the full weight of the frame impacts the head or a portion of the weight as the plaintiff's expert opines.

Sum the moments about the the end of the frame just as it strikes the head. Counter Clockwise and Down is positive.

$$\Sigma T = I \cdot \alpha = I_{cc} \cdot \frac{y''(t)}{h} \qquad \alpha = \frac{y''(t)}{h}$$

$$I_{cc} \cdot \frac{y''(t)}{h} = -R_m(t) \cdot h$$

$$I_{cc} := \frac{w}{3 \cdot g} \cdot \frac{Area_{total}}{(Area_{total} - Area_{in})} \cdot \left[h^2 - \frac{(L - 2 \cdot b_f) \cdot (h - 2 \cdot b_f)}{L \cdot h} \cdot \left(h^2 - h \cdot b_f + b_f^{\,2} \right) \right]$$

$$I_{cc} = 2.11\, s^2 \cdot ft \cdot lb$$

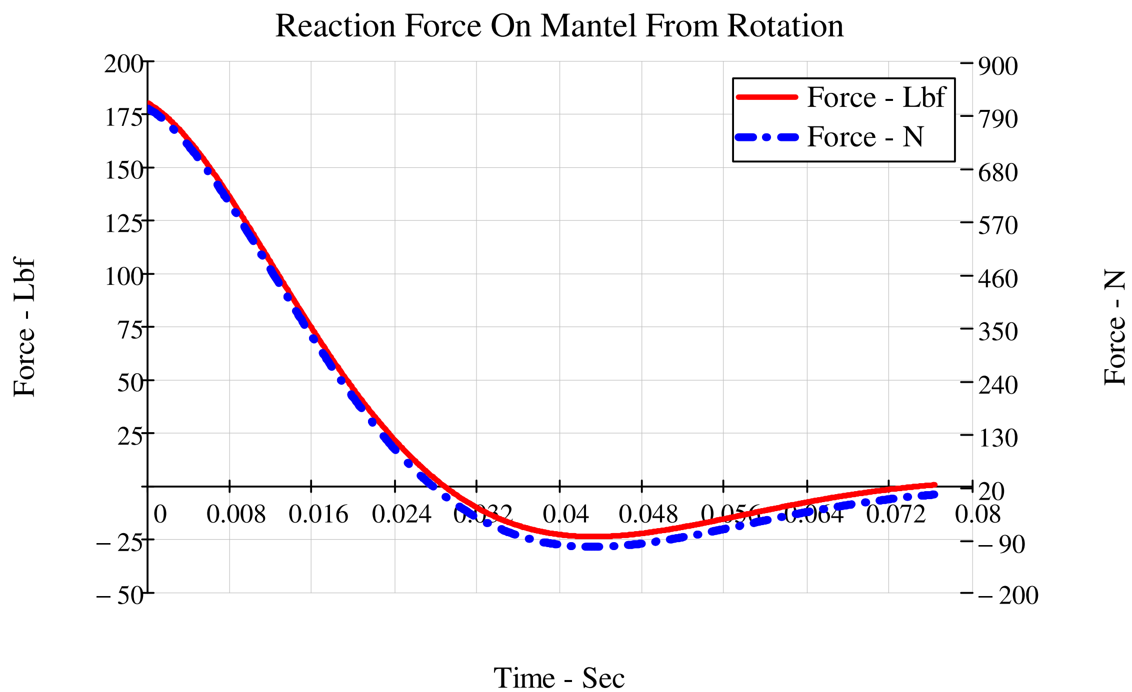
$$R_m(t) := \frac{-I_{cc} \cdot y''(t) \cdot g}{h^2}$$

$R_m(t) =$

$\cdot lbf$
180
180
...

$R_m(t) =$

$\cdot N$
803
801
...



At time = 0 the head reaction is maximum at (-930) N and the mantel reaction is (+803) lbf. At about 44ms the head reaction is about (120) N and the mantel reaction is (-100) N. The spring resists motion. Therefore, the head feels the full weight of the frame at impact because the mantel end is raising up and the force is resisting motion the downward direction.

Up is negative, therefore the mantel end of the frame raises up as the opposite end of the frame strikes the head.

$$y_m(t) := \int_0^T \int_0^T \frac{-h^2 \cdot R_m(t)}{I_{cc} \cdot g} \, dt \, dt$$

$y_m(t) =$

-172
-172
-172
...

$\cdot mm$

REFERENCES:

(1) "Dynamic Characteristics of the Intact, Fused, and Prosthetic-Replaced Cervical Disk", by Michael C. Dahl, Jeffrey P. Rouleau, Stephen Papadopoulos, David J. Nuckley, Randal P. Ching Department of Mechanical Engineering, University of Washington, Seattle, WA 98109

(2) The NATO Research and Technology Organisation (RTO), TR-HFM-090-03.pdf, Chapter 3 – INJURY CRITERIA AND TOLERANCE LEVELS

Based on this study, Mertz [Mertz, 1978] derived two injury tolerance curves based on the upper neck axial compression force measured on a 50th percentile male Hybrid III (see Figure 3.16). The coordinates of the 'Upper' red curve are 0 ms and 6670 N, 35 ms and 1110 N, and greater than 35 ms, 1110 N.

The coordinates of the 'Lower' blue curve are 0 ms and 4000 N, 30 ms and 1110 N, and greater than 30 ms, 1110 N. To evaluate neck load signal, pairs of points (force, duration) are plotted on the graph (shown in Figure G2) with the two injury assessment curves. The points are connected together by a series of straight lines. If any of the line segments lie above the upper curve (red), the neck axial compression force is considered to have the potential to produce serious neck injury. If any of the line segments lie above the lower curve (blue) and below the upper red curve, the potential for neck injury from the axial compressive force is considered less likely. If none of the line exceeds the lower curve, the probability of neck injury from axial compressive force is considered remote.

These levels were proposed for an adult population that was considerably older (exact age range not known) and much less conditioned than a high school football athlete. The time durations were determined from the loading times observed during the experiment, which were on the order of 30 – 40 ms in duration. [3]

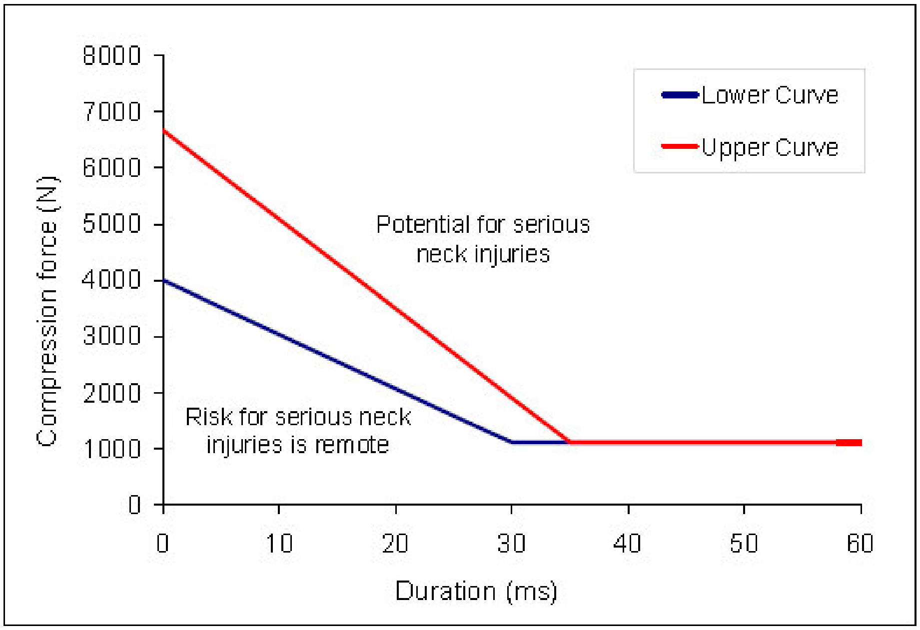


Figure G.2: Injury Tolerance Curves for Axial Neck Compression Force when Using a Hybrid III 50th Percentile Male ATD [Mertz, 1978].

DERIVATIONS

Derivation of the Displacement, Velocity and Acceleration of the head being struck by the frame. Equation of forces on the neck as a result of the frame striking the head. Where **b** is the neck damping; **k** is the neck spring constant; **W** weight of the head; **w** is the weight of the picture frame.

$$M \cdot y'' = -k \cdot y - b \cdot y'$$

$$M \cdot y'' + b \cdot y' + k \cdot y = 0$$

$$y'' + \frac{b}{M} \cdot y' + \frac{k}{M} \cdot y = 0$$

Let: $2\gamma = \frac{b}{M}$ $\qquad \omega_o^2 = \frac{k}{M}$

$$y'' + 2\gamma \cdot y' + \omega_o^2 \cdot y = 0$$

Trial solution: $y(t) = e^{r \cdot t}$, $y'(t) = r \cdot e^{r \cdot t}$, $y''(t) = r^2 \cdot e^{r \cdot t}$

$$r^2 \cdot e^{r \cdot t} + 2\gamma \cdot r \cdot e^{r \cdot t} + \omega^2 e^{r \cdot t} = 0$$

$$r^2 + 2\gamma \cdot r + \omega_o^2 = 0$$

$$r = -\gamma + \sqrt{\gamma^2 - \omega_o^2} \qquad r = -\gamma - \sqrt{\gamma^2 - \omega_o^2}$$

$$r = -\gamma + i \cdot \kappa \qquad r = -\gamma - i \cdot \kappa$$

For real roots let ;

$$\kappa^2 = \gamma^2 - \omega_o^2$$

$$-\kappa^2 = \omega_o^2 - \gamma^2$$

$$i \cdot \kappa = \sqrt{\omega_o^2 - \gamma^2}$$

$$y(t) = C_1 \cdot e^{-\gamma \cdot t + i \cdot \kappa \cdot t} + C_2 \cdot e^{-\gamma \cdot t - i \cdot \kappa \cdot t}$$

$$y(t) = e^{-\gamma \cdot t} \cdot \left(C_1 \cdot e^{i \cdot \kappa \cdot t} + C_2 \cdot e^{-i \cdot \kappa \cdot t}\right)$$

Initial conditions for head movement on the neck; $y(0) = 0$, $y'(0) = v_2$

$$y(t) = e^{-\gamma \cdot t} \cdot \left(C_1 \cdot \cos(\kappa \cdot t) + C_2 \cdot \sin(\kappa \cdot t)\right)$$

$$C_1 = 0$$

$$y(t) = e^{-\gamma \cdot t} \cdot C_2 \cdot \sin(\kappa \cdot t)$$

$$y'(t) = e^{-\gamma \cdot t} \cdot C_2 \cdot \kappa \cdot \cos(\kappa \cdot t) - \gamma \cdot e^{-\gamma \cdot t} \cdot C_2 \cdot \sin(\kappa \cdot t)$$

$$v_2 = \kappa \cdot C_2$$

$$C_2 = \frac{v_2}{\kappa}$$

$$y(t) = \frac{v_2}{\kappa} \cdot e^{-\gamma \cdot t} \cdot \sin(\kappa \cdot t)$$

Calculation of the mass moment of inertia of the frame subtracting the inner area from the overall area of the frame. Where the Mass of the empty area is a function of the ratio of the area of the empty portion to the overall area. The rotation will be about the bottom of the frame I_{cc}.

Find the mass of the inner portion of the frame.

$$M_{total} = M_{frame} + M_{in} \qquad Area_{total} := L \cdot h \qquad Area_{in} := (L - 2b_f)\cdot(h - 2\cdot b_f)$$

$$M_{in} = M_{total}\cdot\frac{Area_{in}}{Area_{total}} \qquad M_{frame} := \frac{w}{g}$$

$$M_{total} = M_{frame} + M_{total}\cdot\frac{Area_{in}}{Area_{total}} \qquad M_{total} = \frac{M_{frame}}{\left(1 - \frac{Area_{in}}{Area_{total}}\right)}$$

$$M_{in} = M_{total} - M_{frame} \qquad M_{in} = \frac{M_{frame}}{\left(1 - \frac{Area_{in}}{Area_{total}}\right)} - M_{frame}$$

$$M_{in} := M_{frame}\cdot\left(\frac{Area_{total}}{Area_{total} - Area_{in}} - 1\right) \qquad M_{in} = 1.248\,\frac{s^2\cdot lb}{ft}$$

$$M_{total} := M_{frame} + M_{in} \qquad M_{total} = 1.907\,\frac{s^2\cdot lb}{ft}$$

$$w_{total} := M_{total}\cdot g \qquad w_{total} = 61.356\,lb \qquad M_{in} = 1.248\,\frac{s^2\cdot lb}{ft}$$

$$w_{total} := w\cdot\left(\frac{Area_{total}}{Area_{total} - Area_{in}}\right) \qquad M_{frame} = 0.659\,\frac{s^2\cdot lb}{ft}$$

$$w_{in} := w_{total} - w$$

$$w := w_{total}\cdot\left(1 - \frac{Area_{in}}{Area_{total}}\right) \qquad w = 21.2\,lb$$

$$I_{cc} = \frac{1}{12}\cdot M_{total}\cdot h^2 + M_{total}\cdot\left(\frac{h}{2}\right)^2 - \left[\frac{1}{12}\cdot M_{in}\cdot\left(h - 2\cdot b_f\right)^2 + M_{in}\cdot\left(\frac{h}{2}\right)^2\right]$$

Apply parallel axis theorm.

$$I_{cc} := \frac{M_{total}}{3}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]$$

$$I_{cc} = 2.11\,s^2\cdot ft\cdot lb$$

Calculation of **ω** the angular velocity of the falling frame. the Potential Energy equals the Linear Kinetic Energy plus the Angular Kinetic Energy. (**w** x **h**/**2**) = 1/2 x **w**/**g** x $\mathbf{v}^2$ + 1/2 x $\mathbf{I_{cc}}$ x $\boldsymbol{\omega}^2$

$$w\cdot\frac{h}{2} = \frac{1}{2}\cdot\frac{w}{g}\cdot\left(\omega\cdot\frac{h}{2}\right)^2 + \frac{1}{2}\cdot I_{cc}\cdot\omega^2$$

$$w\cdot\frac{h}{2} = \frac{1}{2}\cdot\frac{w}{g}\cdot\frac{h^2}{4}\cdot\omega^2 + \frac{1}{2}\cdot\left[\frac{M_{total}}{3}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

$$w\cdot h = \frac{w}{g}\cdot\frac{h^2}{4}\cdot\omega^2 + \left[\frac{w_{total}}{3g}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

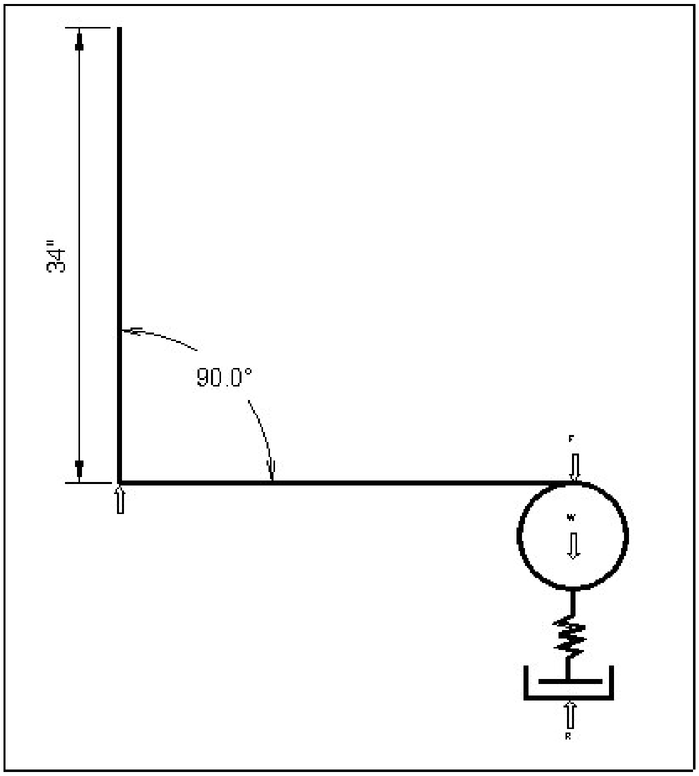
$$h = \frac{h^2}{4\cdot g}\cdot\omega^2 + \left[\frac{Area_{total}}{3\cdot g\left(Area_{total} - Area_{in}\right)}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

$$h = \left[\frac{h^2}{4\cdot g} + \frac{Area_{total}}{3\cdot g\left(Area_{total} - Area_{in}\right)}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

$$12\cdot g\cdot h = \left[3h^2 + \frac{4Area_{total}}{\left(Area_{total} - Area_{in}\right)}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

$$\omega = \sqrt{\frac{12 \cdot g \cdot h}{\left[3h^2 + \frac{4Area_{total}}{\left(Area_{total} - Area_{in}\right)} \cdot \left[h^2 - \frac{\left(L - 2 \cdot b_f\right) \cdot \left(h - 2 \cdot b_f\right)}{L \cdot h} \cdot \left(h^2 - h \cdot b_f + b_f^{\,2}\right)\right]\right]}}$$

This is the **ω** for the frame as it rotated to approximately 90 degrees (horizontal) of its rotation.



Frame rotates and falls on head.

The free body diagram of the frame striking the head where the neck acts like a spring with damping.

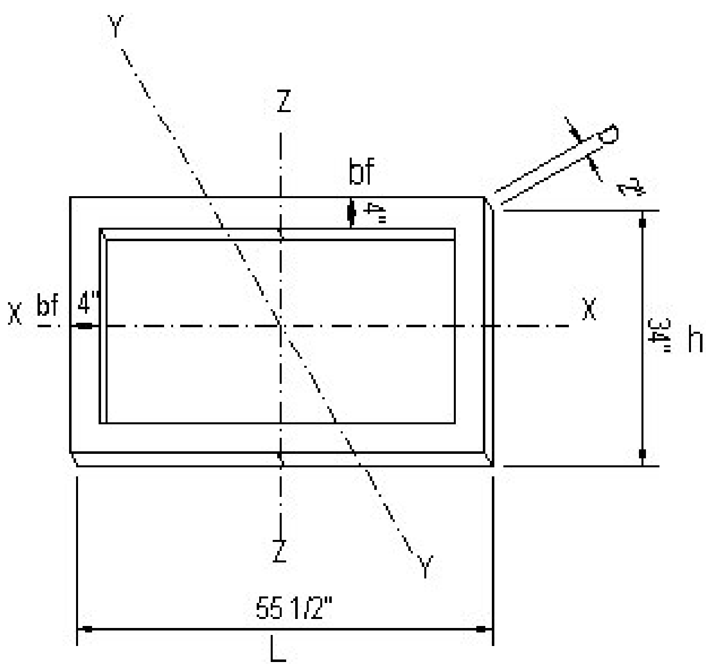


Diagram of the frame that strikes head.

Frame dimensions.

Picture frame falls on mantel, rotates and strikes the head of woman sitting on an ottaman. The frame falls directly down (vertically) strikes a fireplace mantel and rotates approximatley 90 degrees to strike woman on top of her head.

Wt= weight of head = 9.8 lbs
w = weight of frame = 21.2 lbs
L = length of frame = 55-1/2 inches
h = height of frame = 34 inches
b_f = Frame width = 4 inches
k = Spring Constants for the neck = 425000 N/m
b = Damping for the neck = 1500 N/m/s

Potential Energy = Kinetic Energy + Rotational Energy
$w \times h/2 = 1/2 \times w/g \times \omega^2 \times (h/2)^2 + 1/2 \times I_{cc} \times \omega^2$

$L := 55.5\text{in} \qquad Wt := 9.8\text{lb} \qquad w := 21.2\text{lb} \qquad h := 34\text{in} \qquad b_f := 4\text{in}$

$$\text{Area}_{total} := L \cdot h \qquad \text{Area}_{in} := (L - 2b_f)\cdot(h - 2\cdot b_f) \qquad w_{total} := w\cdot\left(\frac{\text{Area}_{total}}{\text{Area}_{total} - \text{Area}_{in}}\right)$$

$$k := \frac{425000\frac{N}{m}}{g} \qquad k = 29122\frac{lb}{ft}$$

Spring stiffness of neck per reference note (1).

$$M := \frac{Wt}{g} \qquad \omega_o := \sqrt{\frac{k}{M}} \qquad \omega_o = 309.2\frac{1}{s}$$

Natural frequency ω_o of the head.

$$\omega := \sqrt{\frac{12\cdot g\cdot h}{\left[3h^2 + \frac{4\text{Area}_{total}}{(\text{Area}_{total} - \text{Area}_{in})}\cdot\left[h^2 - \frac{(L - 2\cdot b_f)\cdot(h - 2\cdot b_f)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]}}$$

$$\omega = 4.18\frac{1}{s}$$

Angular velocity of frame as it rotates through $\pi/2$.

$$v := \omega\cdot\left(h - \frac{b_f}{2}\right) \qquad v = 11.2\frac{ft}{s}$$

Velocity that frame strikes head.

$$v_2 := v\cdot\frac{w}{w + Wt} \qquad v_2 = 7.6\frac{ft}{s}$$

Velocity of head after impact using Momentum analysis.

$b := \frac{1500}{g} \frac{N \cdot s}{m}$ $\quad b = 102.8 \frac{s \cdot lb}{ft}$ Damping constant of neck per.

$\gamma := \frac{b}{2 \cdot M}$ $\quad \gamma = 168.7 \frac{1}{s}$ $\quad T := \frac{2 \cdot \pi}{\omega_o}$ $\quad T = 0.02 s$

$f := \frac{1}{T}$ $\quad f = 49.2 \cdot Hz$

$\kappa := \sqrt{\omega_o^2 - \gamma^2}$ $\quad \kappa = 259.1 \frac{1}{s}$ $\quad t := 0ms, 0.1ms .. T$ The contact time in milliseconds as the frame strikes the head to determine the force in g's.

Mathematical differential equation solution of head on spring being struck with force and compressing the spring (neck).

$y(t) := \frac{v_2}{\kappa} \cdot e^{-\gamma \cdot t} \cdot \sin(\kappa \cdot t)$ $\quad y'(t) := \left(\frac{d}{dt} y(t)\right)$ $\quad y''(t) := \left(\frac{d^2}{dt^2} y(t)\right)$

$y(t) =$ (·mm)

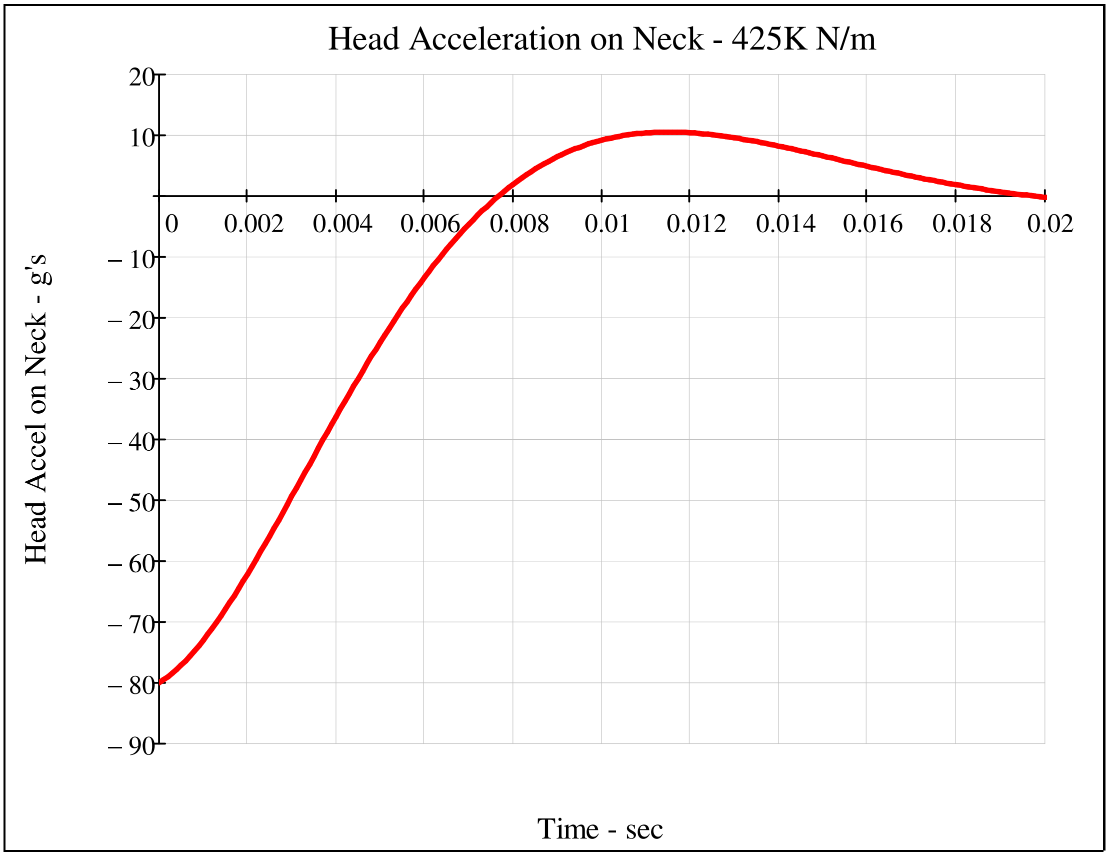
0
0.23
0.45
0.66
0.87
1.07
...

$y'(t) =$ ($\frac{ft}{s}$)

7.6
7.4
7.1
6.9
6.6
6.4
...

$y''(t) =$ (·g)

-80
-79.5
-79
-78.4
-77.8
-77.1
...


Head Acceleration on Neck - 425K N/m
Head Accel on Neck - g's
20
10
– 10
– 20
– 30
– 40
– 50
– 60
– 70
– 80
– 90
0
0.002
0.004
0.006
0.008
0.01
0.012
0.014
0.016
0.018
0.02
Time - sec

$$force(t) := \frac{Wt}{g} \cdot y''(t)$$

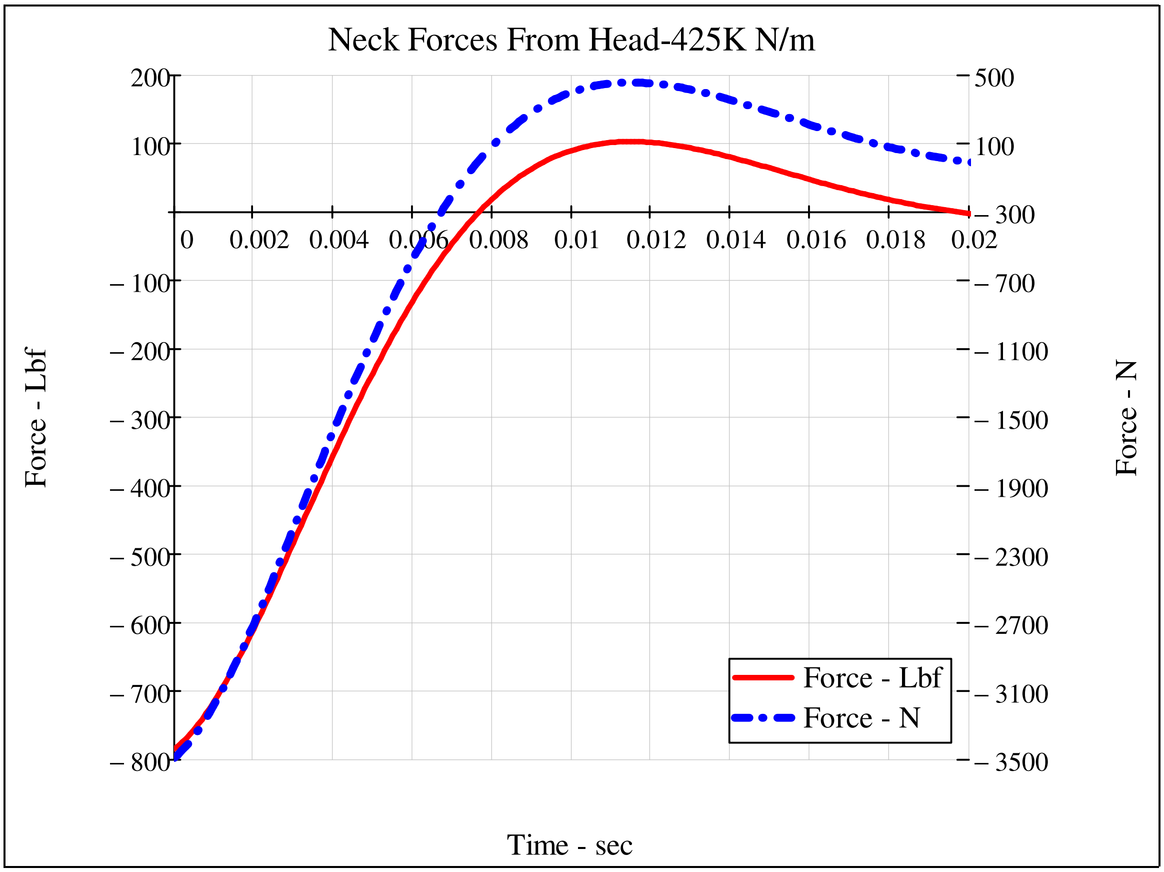
The force that the neck feels due to the acceleration of the head using Newton's 2nd Law; Force = Mass x Acceletation
F = M x a
Where acceleration = y''(t)

$force(t) \cdot g =$

-784	·lbf
-780	
-774	
-769	
...	

$force(t) \cdot g =$

-3488	·N
-3468	
-3445	
-3420	
...	



The Green line represents the Impulse of the frame/head strike on the neck as calculated.

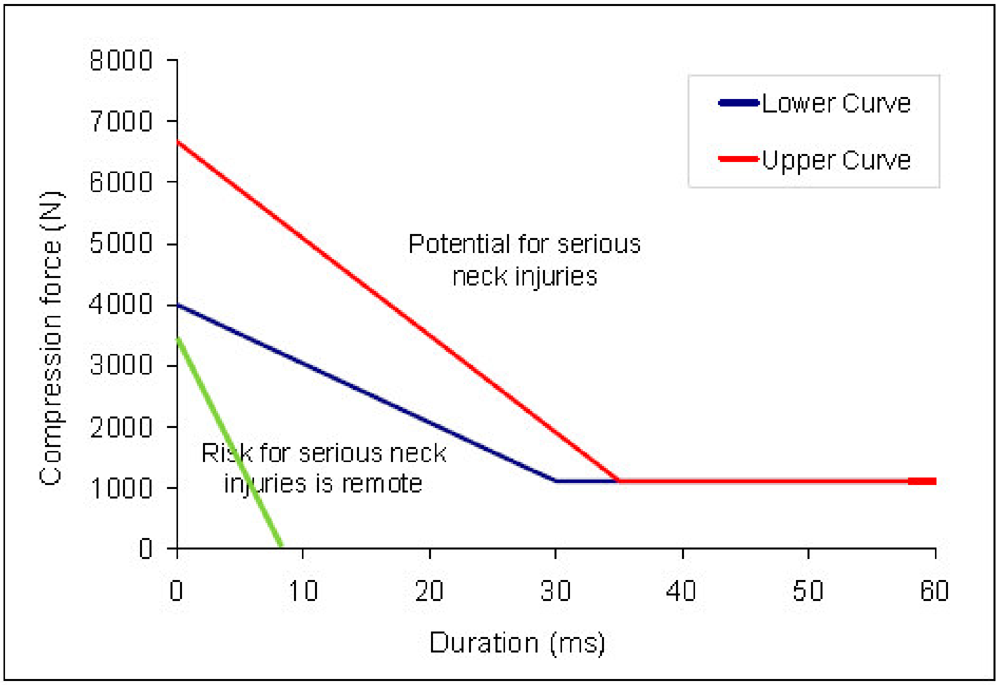


Figure G.2: Injury Tolerance Curves for Axial Neck Compression Force when Using a Hybrid III 50th Percentile Male ATD [Mertz, 1978].

The following analysis is to calculate the impulse and delta-V values that the neck feels as a result of the frame inpact.

$t_1 := 0s$ $t_2 := 0.0074s$

$$\text{Impulse} := \int_{t_1}^{t_2} \text{force}(t)\,dt \cdot g \qquad \text{Impulse} = -2.96\,s\cdot lbf$$

$$\text{deltaV} := \frac{\text{Impulse}}{\text{Wt}} \qquad \text{deltaV} = -9.7\frac{1}{s}\cdot ft \qquad \frac{y''(0)}{g} = -80$$

This analysis is to determine if the full weight of the frame impacts the head or a portion of the weight as the plaintiff's expert opines.

Sum the moments about the the end of the frame just as it strikes the head. Counter Clockwise and Down is positive.

$$\Sigma T = I \cdot \alpha = I_{cc} \cdot \frac{y''(t)}{h} \qquad \alpha = \frac{y''(t)}{h}$$

$$I_{cc} \cdot \frac{y''(t)}{h} = -R_m(t) \cdot h$$

$$I_{cc} := \frac{w}{3 \cdot g} \cdot \frac{Area_{total}}{\left(Area_{total} - Area_{in}\right)} \cdot \left[h^2 - \frac{\left(L - 2 \cdot b_f\right) \cdot \left(h - 2 \cdot b_f\right)}{L \cdot h} \cdot \left(h^2 - h \cdot b_f + b_f^{\,2}\right)\right]$$

$$I_{cc} = 2.11\, s^2 \cdot ft \cdot lb$$

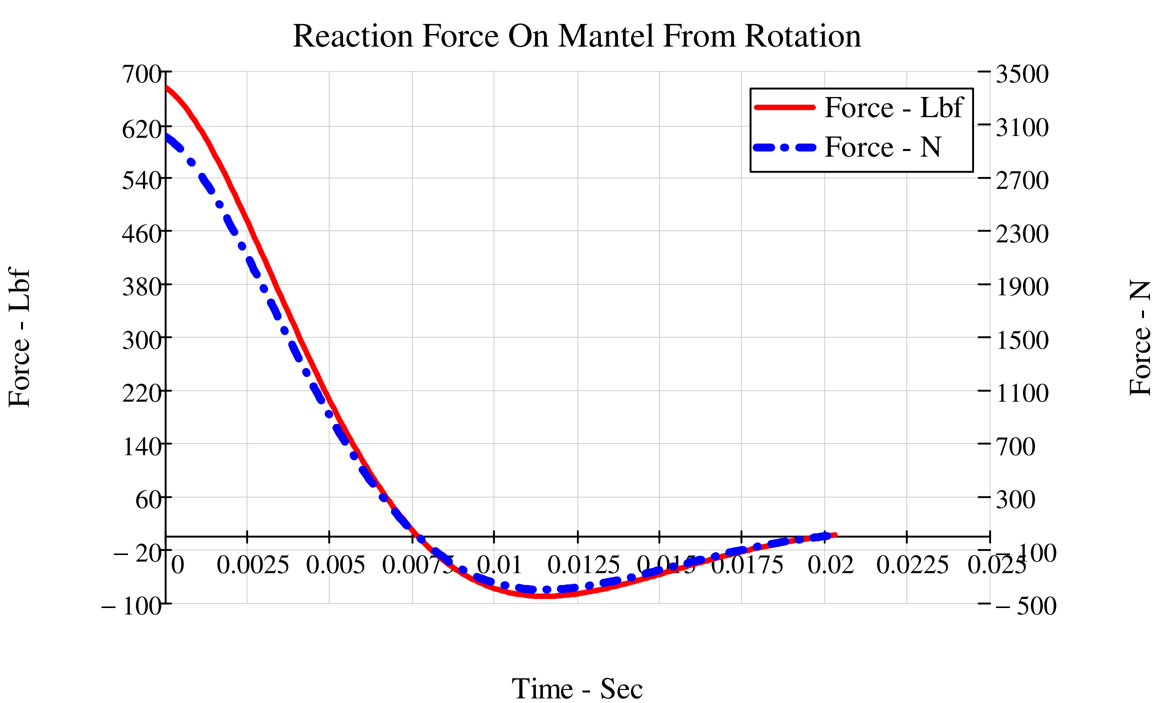
$$R_m(t) := \frac{-I_{cc} \cdot y''(t) \cdot g}{h^2}$$

$R_m(t) =$ (lbf)

677
673
...

$R_m(t) =$ (N)

3010
2992
...



At time = 0 the head reaction is maximum at (-3488) Newtons and the mantel reaction is (+3010) Newtons. At about 11ms the head reaction is about (+490) Newtons and the mantel reaction is (-400) Newtons. The spring resists motion. Therefore, the head feels the full weight of the frame at impact because the mantel end is raising up and the force is resisting motion the downward direction.

Up is negative, therefore the mantel end of the frame raises up as the opposite end of the frame strikes the head.

$$y_m(t) := \int_0^T \int_0^T \frac{-h^2 \cdot R_m(t)}{I_{cc} \cdot g} \, dt \, dt \qquad y_m(t) = \begin{bmatrix} -46 \\ -46 \\ -46 \\ \dots \end{bmatrix} \cdot mm$$

REFERENCES:

(1) "Dynamic Characteristics of the Intact, Fused, and Prosthetic-Replaced Cervical Disk", by Michael C. Dahl, Jeffrey P. Rouleau, Stephen Papadopoulos, David J. Nuckley, Randal P. Ching Department of Mechanical Engineering, University of Washington, Seattle, WA 98109

(2) The NATO Research and Technology Organisation (RTO), TR-HFM-090-03.pdf, Chapter 3 – INJURY CRITERIA AND TOLERANCE LEVELS

Based on this study, Mertz [Mertz, 1978] derived two injury tolerance curves based on the upper neck axial compression force measured on a 50_{th} percentile male Hybrid III (see Figure 3.16). The coordinates of the 'Upper' curve are 0 ms and 6670 N, 35 ms and 1110 N, and greater than 35 ms, 1110 N.

The coordinates of the 'Lower' curve are 0 ms and 4000 N, 30 ms and 1110 N, and greater than 30 ms, 1110 N. To evaluate neck load signal, pairs of points (force, duration) are plotted on the graph (shown in Figure 3.16) with the two injury assessment curves. The points are connected together by a series of straight lines. If any of the line segments lie above the upper curve (red), the neck axial compression force is considered to have the potential to produce serious neck injury. If any of the line segments lie above the lower curve (blue), the potential for neck injury from the axial compressive force is considered less likely.
If none of the line exceeds the lower curve, the probability of neck injury from axial compressive force is considered remote.

These levels were proposed for an adult population that was considerably older (exact age range not known) and much less conditioned than a high school football athlete. The time durations were determined from the loading times observed during the experiment, which were on the order of 30 – 40 ms. [3]

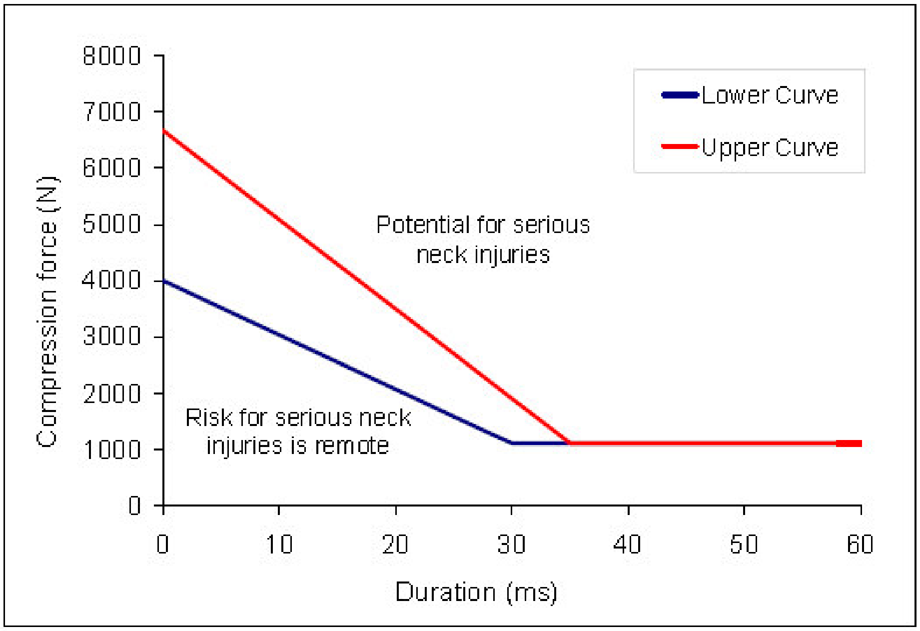


Figure G.2: Injury Tolerance Curves for Axial Neck Compression Force when Using a Hybrid III 50th Percentile Male ATD [Mertz, 1978].

DERIVATIONS

Derivation of the Displacement, Velocity and Acceleration of the head being struck by the frame. Equation of forces on the neck as a result of the frame striking the head. Where **b** is the neck damping; **k** is the neck spring constant; **W** weight of the head; **w** is the weight of the picture frame.

$$M \cdot y'' = -k \cdot y - b \cdot y'$$

$$M \cdot y'' + b \cdot y' + k \cdot y = 0$$

$$y'' + \frac{b}{M} \cdot y' + \frac{k}{M} \cdot y = 0$$

Let: $2\gamma = \frac{b}{M}$ $\qquad \omega_o^{\ 2} = \frac{k}{M}$

$$y'' + 2\gamma \cdot y' + \omega_o^{\ 2} \cdot y = 0$$

Trial solution: $y(t) = e^{r \cdot t}$

$y'(t) = r \cdot e^{r \cdot t}$

$y''(t) = r^2 \cdot e^{r \cdot t}$

$$r^2 \cdot e^{r \cdot t} + 2\gamma \cdot r \cdot e^{r \cdot t} + \omega^2 e^{r \cdot t} = 0$$

$$r^2 + 2\gamma \cdot r + \omega_o^{\ 2} = 0$$

$$r = -\gamma + \sqrt{\gamma^2 - \omega_o^{\ 2}} \qquad r = -\gamma - \sqrt{\gamma^2 - \omega_o^{\ 2}}$$

$$r = -\gamma + i \cdot \kappa \qquad r = -\gamma - i \cdot \kappa$$

For real roots let ;

$\kappa^2 = \gamma^2 - \omega_o^{\ 2}$

$-\kappa^2 = \omega_o^{\ 2} - \gamma^2$

$i \cdot \kappa = \sqrt{\omega_o^{\ 2} - \gamma^2}$

$$y(t) = C_1 \cdot e^{-\gamma \cdot t + i \cdot \kappa \cdot t} + C_2 \cdot e^{-\gamma \cdot t - i \cdot \kappa \cdot t}$$

$$y(t) = e^{-\gamma \cdot t} \cdot \left(C_1 \cdot e^{i \cdot \kappa \cdot t} + C_2 \cdot e^{-i \cdot \kappa \cdot t}\right)$$

Initial conditions for head movement on the neck; $y(0) = 0$ $\qquad y'(0) = v_2$

$$y(t) = e^{-\gamma \cdot t} \cdot \left(C_1 \cdot \cos(\kappa \cdot t) + C_2 \cdot \sin(\kappa \cdot t)\right)$$

$$C_1 = 0$$

$$y(t) = e^{-\gamma \cdot t} \cdot C_2 \cdot \sin(\kappa \cdot t)$$

$$y'(t) = e^{-\gamma \cdot t} \cdot C_2 \cdot \kappa \cdot \cos(\kappa \cdot t) - \gamma \cdot e^{-\gamma \cdot t} \cdot C_2 \cdot \sin(\kappa \cdot t)$$

$$v_2 = \kappa \cdot C_2$$

$$C_2 = \frac{v_2}{\kappa}$$

$$y(t) = \frac{v_2}{\kappa} \cdot e^{-\gamma \cdot t} \cdot \sin(\kappa \cdot t)$$

Calculation of the mass moment of inertia of the frame subtracting the inner area from the overall area of the frame. Where the Mass of the empty area is a function of the ratio of the area of the empty portion to the overall area. The rotation will be about the bottom of the frame I_{cc}.

Find the mass of the inner portion of the frame.

$$M_{total} = M_{frame} + M_{in} \qquad Area_{total} := L \cdot h \qquad Area_{in} := (L - 2b_f) \cdot (h - 2 \cdot b_f)$$

$$M_{in} = M_{total} \cdot \frac{Area_{in}}{Area_{total}} \qquad M_{frame} := \frac{w}{g}$$

$$M_{total} = M_{frame} + M_{total} \cdot \frac{Area_{in}}{Area_{total}} \qquad M_{total} = \frac{M_{frame}}{\left(1 - \frac{Area_{in}}{Area_{total}}\right)}$$

$$M_{in} = M_{total} - M_{frame} \qquad M_{in} = \frac{M_{frame}}{\left(1 - \frac{Area_{in}}{Area_{total}}\right)} - M_{frame}$$

$$M_{in} := M_{frame} \cdot \left(\frac{Area_{total}}{Area_{total} - Area_{in}} - 1\right) \qquad M_{in} = 1.248 \frac{s^2 \cdot lb}{ft}$$

$$M_{total} := M_{frame} + M_{in} \qquad M_{total} = 1.907 \frac{s^2 \cdot lb}{ft}$$

$$w_{total} := M_{total} \cdot g \qquad w_{total} = 61.356 lb \qquad M_{in} = 1.248 \frac{s^2 \cdot lb}{ft}$$

$$w_{total} := w \cdot \left(\frac{Area_{total}}{Area_{total} - Area_{in}}\right) \qquad M_{frame} = 0.659 \frac{s^2 \cdot lb}{ft}$$

$$w_{in} := w_{total} - w$$

$$w := w_{total} \cdot \left(1 - \frac{Area_{in}}{Area_{total}}\right) \qquad w = 21.2 lb$$

$$I_{cc} = \frac{1}{12}\cdot M_{total}\cdot h^2 + M_{total}\cdot\left(\frac{h}{2}\right)^2 - \left[\frac{1}{12}\cdot M_{in}\cdot\left(h - 2\cdot b_f\right)^2 + M_{in}\cdot\left(\frac{h}{2}\right)^2\right]$$

Apply parallel axis theorm.

$$I_{cc} := \frac{M_{total}}{3}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]$$

$$I_{cc} = 2.11\, s^2\cdot ft\cdot lb$$

Calculation of **ω** the angular velocity of the falling frame. the Potential Energy equals the Linear Kinetic Energy plus the Angular Kinetic Energy. (**w** x **h/2**) = 1/2 x **w/g** x $\mathbf{v}^2$ + 1/2 x $\mathbf{I_{cc}}$ x $\boldsymbol{\omega}^2$

$$w\cdot\frac{h}{2} = \frac{1}{2}\cdot\frac{w}{g}\cdot\left(\omega\cdot\frac{h}{2}\right)^2 + \frac{1}{2}\cdot I_{cc}\cdot\omega^2$$

$$w\cdot\frac{h}{2} = \frac{1}{2}\cdot\frac{w}{g}\cdot\frac{h^2}{4}\cdot\omega^2 + \frac{1}{2}\cdot\left[\frac{M_{total}}{3}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

$$w\cdot h = \frac{w}{g}\cdot\frac{h^2}{4}\cdot\omega^2 + \left[\frac{w_{total}}{3g}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

$$h = \frac{h^2}{4\cdot g}\cdot\omega^2 + \left[\frac{Area_{total}}{3\cdot g\left(Area_{total} - Area_{in}\right)}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

$$h = \left[\frac{h^2}{4\cdot g} + \frac{Area_{total}}{3\cdot g\left(Area_{total} - Area_{in}\right)}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

$$12\cdot g\cdot h = \left[3h^2 + \frac{4Area_{total}}{\left(Area_{total} - Area_{in}\right)}\cdot\left[h^2 - \frac{\left(L - 2\cdot b_f\right)\cdot\left(h - 2\cdot b_f\right)}{L\cdot h}\cdot\left(h^2 - h\cdot b_f + b_f^{\,2}\right)\right]\right]\cdot\omega^2$$

$$\omega = \sqrt{\frac{12 \cdot g \cdot h}{\left[3h^2 + \frac{4Area_{total}}{\left(Area_{total} - Area_{in}\right)} \cdot \left[h^2 - \frac{\left(L - 2 \cdot b_f\right) \cdot \left(h - 2 \cdot b_f\right)}{L \cdot h} \cdot \left(h^2 - h \cdot b_f + b_f^{\,2}\right)\right]\right]}}$$

This is the **ω** for the frame as it rotated to approximately 90 degrees (horizontal) of its rotation.